<u>Def</u>. Let Σ be any set. A <u>discrete-time dynamical system</u> on Σ is given by a function $f : \Sigma \circlearrowleft$.
If $x_0 \in \Sigma$, then the sequence on Σ

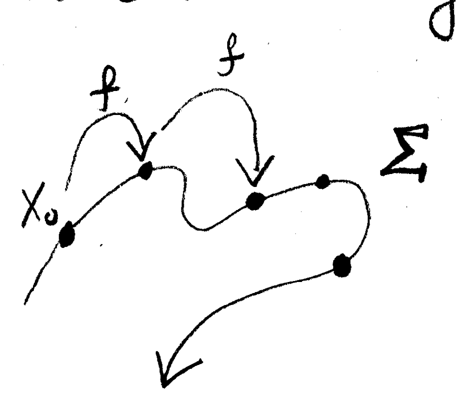
$$x_0 \curvearrowright x_1 \curvearrowright x_2, \ldots$$

where
$$x_{t+1} = f(x_t) \qquad (2)$$
is called the (forward) orbit through x_0.

<u>Notation</u> $f^t(x) \stackrel{\Delta}{=} f(f^{t-1}(x))$ with $f^0(x) = x \Rightarrow f^1 = f$.
and $x_t = f^t(x_0)$.

<u>REMARKS</u>

- Σ and f are completely arbitrary.
- The system (1) always defines a discrete-time dynamical system on $\Sigma = \mathbb{R}^N$, where f is the function mapping any $X_0 \in \mathbb{R}^N$ to $X(1)$ where $X(t)$ is the orbit through X_0.

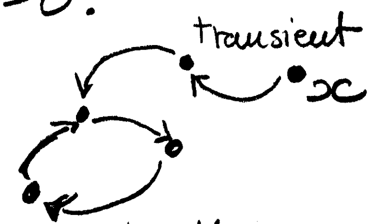


- If f is one-to-one & onto, then the dynamical system (2) is <u>invertible</u>, i.e., every point has a backward orbit as well

$$\cdots\ x_{-2} \xleftarrow{f^{-1}} x_{-1} \xleftarrow{f^{-1}} x_0 \xrightarrow{f} x_1 \quad x_2 \cdots$$

Def A point x is a fixed point if $x = f(x)$.
The orbit through x is periodic with period n ("n-cycle") if $f^n(x) = x$. The smallest +ve period is called the minimal period.

A point x is eventually periodic if $f^t(x)$ is periodic for some $t > 0$.



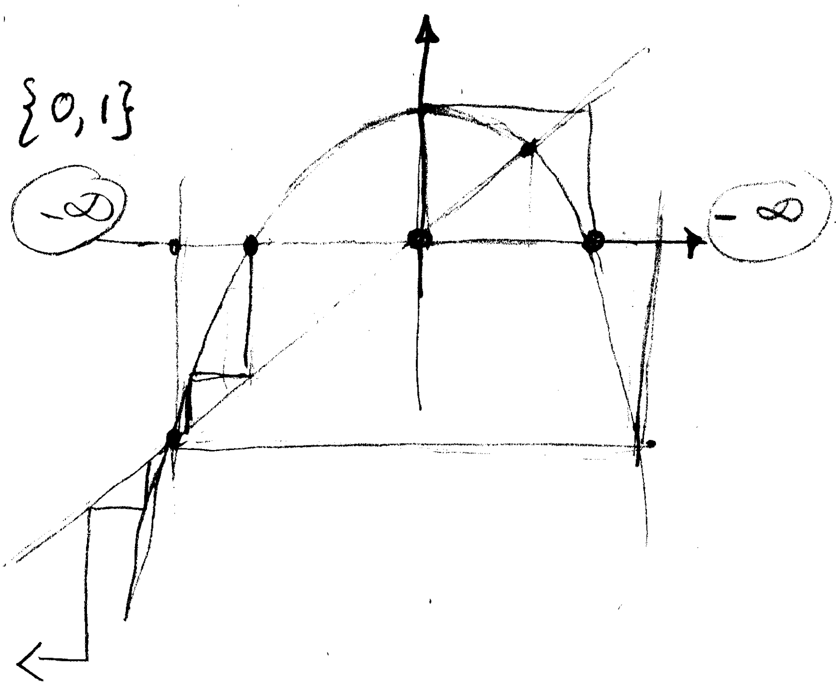
Ex Let $\Sigma = \mathbb{R}$ and $f(x) = 1 - x^2$

Fixed points: $f(x) = x \Rightarrow x^2 + x - 1 = 0$

$$x = \frac{-1 \mp \sqrt{5}}{2}$$

$f(0) = 1$, $f(1) = 0$ so $\{0, 1\}$ is a 2-cycle

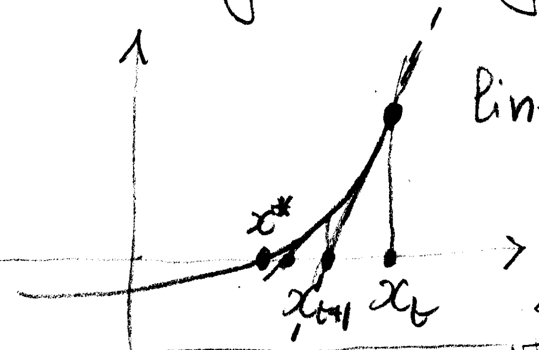
$f(-1) = 0$, so -1 is eventually periodic and so is $\frac{1+\sqrt{5}}{2}$.



Define $P_n(x) = f^n(x) - x$. Then the solutions of the equation $P_n(x) = 0$ are the d-cycles, for some divisor d of n.

Ex The Newton's method.

Solve $g(x)=0$ $\quad g: \mathbb{R}^{\circlearrowleft}$ differentiable



line Tangent To g at x_t:

$$y - g(x_t) = g'(x_t)(x - x_t)$$

let $y=0$ & solve for $x = x_{t+1}$:

$$\boxed{x_{t+1} = x_t - \frac{g(x_t)}{g'(x_t)} = f(x_t)}$$

a discrete time dynamical system!

$$f(x^*) = x^* - \frac{g(x^*)}{g'(x^*)}$$

Now $g(x^*)=0$, so if $g'(x^*) \neq 0$ then $f(x^*)=x^*$

Conversely, if $f(x^*)=x^* \Rightarrow \frac{g(x^*)}{g'(x^*)} = 0$

So if $g'(x)=0$ then $g(x)=0$ iff x is a fixed point of the Newton's method.

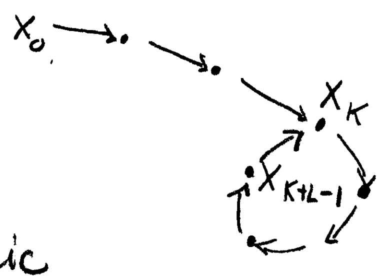
Lemma: Let f be invertible. Then every pre-periodic point is periodic.

Proof: x_0 pre-periodic $\Rightarrow \exists\, K, L$ with $x_{K+L} = x_K \Rightarrow$

$$f^{K+L}(x_0) = f^K(x_0)$$

$$(f^{-1})^K(f^{K+L}(x_0)) = (f^{-1})^K f^K(x_0)$$

or $f^L(x_0) = x_0 \Rightarrow x_0$ is periodic



$\square$

Def A set A is an <u>attractor</u> for f, if there exists a neighbourhood U of A and a positive integer N such that $f^N(U) \subset U$ and

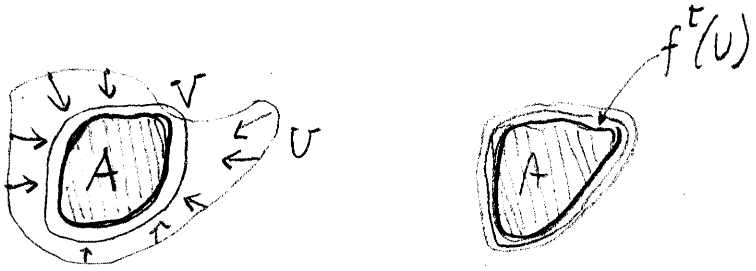
$$A = \bigcap_{t=1}^{\infty} f^t(U)$$

The set U is called a <u>fondamental neighbourhood</u> of A.

An attractor is usually required to be compact ($\sim$ close & bounded).

Def. The <u>basin of attraction</u> of A is the open set

$$\bigcup_{t>0} (f^t)^{-1}(U)$$



Ex Let $f_\lambda(x) = \lambda x$ $\quad$ $A = \{0\}$ is invariant.

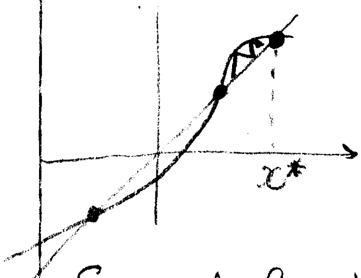
$x_{t+1} = \lambda x_t$ gives $x_t = \lambda^t x_0$. For $|\lambda| < 1$ we have $x_t \to 0$, so $\{0\}$ is an attractor (with $U = (-1, 1)$, say). The basin of attr. is $\mathbb{R}$.

When $|\lambda| = 1$ all pts are periodic (fixed pts for $\lambda = 1$ and 2-cycles for $\lambda = -1$). When $|\lambda| > 1$ the origin is a repeller.

When $\lambda = 0$ all pts are eventually fixed.

1-Dimensional dynamics

Stability of n-cycles. We deal with fixed pts first.



Let $f(x^*) = x^*$ and f smooth at x^*.

Let $x_t = x^* + \delta_t \quad |\delta_t| \ll 1$

Expand f in Taylor series at x^*:

$$x_{t+1} = f(x_t) = f(x^* + \delta_t) = \underbrace{f(x^*)}_{x^*} + f'(x^*)\delta_t + \frac{1}{2} f''(x^*)\delta_t^2 + O(\delta_t^3)$$

$$\delta_{t+1} + \cancel{x^*} = \cancel{x^*} + f'(x^*)\delta_t + \frac{1}{2} f''(x^*)\delta_t^2 + O(\delta_t^3)$$

This gives

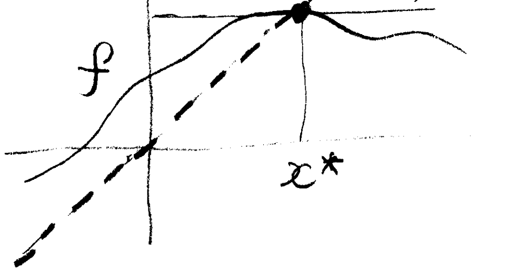
$$\delta_{t+1} = f'(x^*)\delta_t + O(\delta_t^2)$$

and so x^* is an attractor if $|f'(x^*)| < 1$,

If $f'(x^*) = 0$ the rate of convergence to the attractor is faster, being determined by higher-order (typically quadratic) terms

$$\delta_{t+1} = \frac{1}{2} f''(x^*)\delta_t^2 + O(\delta_t^3)$$

the attractor is superstable, i.e., the number of common digits of x_t and x^* doubles at each iteration.



a superstable fixed point

If $|f'(x^*)| = 1$ we most consider higher-order terms.

Let k be the smallest integer greater than 1 for which $\frac{d^k f(x^*)}{d x^k} \neq 0$. (We may assume that such integer exists, lest f is linear and the problem is trivial.)

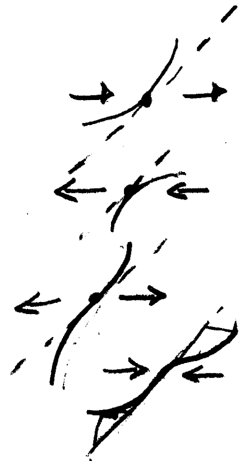
Then we have

i) $f'(x^*) = 1$

$$\delta_{t+1} = \delta_t + \frac{1}{k!} f^{(k)}(x^*)\,\delta_t^k + O(\delta_t^{k+1})$$
$$= \delta_t \underbrace{\left(1 + \frac{1}{k!} f^{(k)}(x^*)\,\delta_t^{k-1} + O(\delta_t^k)\right)}_{\Lambda}$$

4 cases:

k	$f^{(k)}(x^*)$	Λ
even	>0	$\gtrless 1$
even	<0	$\gtrless 1$
odd	>0	>1
odd	<0	<1



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ii) $f'(x^*) = -1$

$$\delta_{t+1} = \delta_t\left(-1 + \frac{1}{k!} f^{(k)}(x^*)\,\delta_t^{k-1} + O(\delta_t^k)\right)$$

k	$f^{(k)}(x^*)$	Λ
odd	>0	>-1
odd	<0	<-1

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For k even, one must look at higher-order terms.

Stability of n-cycles

x^* belongs to an n-cycle of f $\Leftrightarrow$ x^* is a fixed pt of f^n, hence

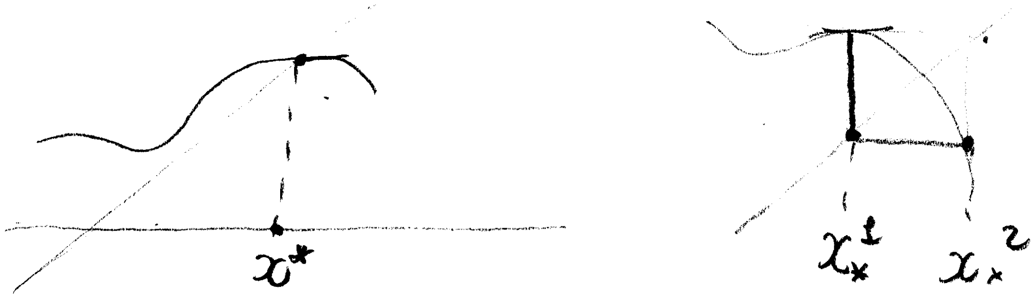
$$|(f^n)'(x^*)| < 1 \iff \text{the n-cycle containing } x^* \text{ is an attractor.}$$

Let $\{x_1^*, x_2^*, \ldots, x_n^*\}$ be an n-cycle. The chain rule of differentiation then gives

$$(f^n)'(x_1^*) = (f^{n-1} \circ f)'(x_1^*) = f'(x_1^*)(f^{n-1})'(x_2^*)$$
$$= f'(x_1^*) f'(x_2^*) (f^{n-2})'(x_3^*) = \prod_{t=1}^{n} f'(x_t^*).$$

It follows that the derivative of f^n is the same at every point of the cycle, and is called the multiplier of the cycle.

In particular, an n-cycle is superstable if $f'(x)$ vanishes at one of its points.
A point x for which $f'(x) = 0$ is called a critical point.



superstable cycles.