

# Scaling dynamics of a cubic interval-exchange transformation

J. H. Lowenstein<sup>†</sup> and F. Vivaldi

<sup>†</sup>*Dept. of Physics, New York University, 2 Washington Place, New York, NY 10003, USA*  
*School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK*

## Abstract

We study the dynamics of renormalization of a specific interval exchange transformation which features exact scaling (the cubic Arnoux-Yoccoz model). Using a symbolic space that describes both dynamics and scaling, we characterize the periodic points of the scaling map in terms of generalized decimal expansions, where the base is the reciprocal of a Pisot number and the digits are algebraic integers. The set of periodic points has a rich arithmetic and geometric structure: we establish rigorously some facts, and use extensive numerical experimentation to formulate a conjecture.

June 23, 2008

## 1 Introduction

There are symbolic representations of dynamical systems which lead to interesting arithmetical phenomena. In such *arithmetic codings*, the symbols represent generalized ‘digits’ in a number system which naturally describes the dynamics. For a survey of this rich area of research from a dynamical perspective, see [29]; an arithmetical viewpoint is developed in [1–3].

In arithmetic codings—even in the simplest settings—one encounters at once interesting problems. A well-known example is the doubling map of the unit interval (circle)  $\gamma : x \mapsto 2x \pmod{1}$ , with the symbolic dynamics defined by the partition of the interval into two equal halves. The symbols representing an orbit are just the binary digits of its initial point, so that the eventually periodic symbolic sequences correspond to the rational numbers, and the periodic sequences to the rationals with odd denominator. For each  $n$ , the set of  $n$ -periodic points

$$\text{Fix}(\gamma^n) = \frac{1}{2^n - 1} \mathbb{Z} \cap [0, 1) \quad n \geq 1 \quad (1)$$

is a lattice. The exponentially large denominator determines the maximal complexity (height) an  $n$ -cycle can have.

The question of minimal complexity is altogether more difficult. When  $2^n - 1$  has a large number of divisors, cancellation becomes likely, leading to  $n$ -cycles with a potentially much smaller denominator. This phenomenon is relevant to the twin asymptotic problem that corresponds to ordering the cycles by increasing (odd) denominator, rather than increasing period. As the denominator increases, the low-complexity cycles are those that are found first. The period of the orbits is determined by the multiplicative order of the integer 2—the constant defining the map—in the group of integers modulo the denominator. The computation then reduces to the case of prime denominators, whose asymptotics are connected to a well-known open problem in number theory, the so-called Artin’s conjecture on primitive roots [27]. At present, the average growth rate of the period of a rational point of the doubling map, as the denominator increases, can only be established assuming the validity of the generalized Riemann hypothesis [22].

Besides the doubling map, there are many other dynamical systems whose periodic behaviour is determined by modular multiplication in an appropriate domain, leading to similar asymptotic problems. A much-studied example is that of hyperbolic toral automorphisms [5, 11, 24]. In place of  $\mathbb{Q}$ , we now have the field  $\mathbb{Q}(\lambda)$ , where  $\lambda$  is an eigenvalue of the automorphism (whose specific choice is immaterial), while the lattice (1), which is a  $\mathbb{Z}$ -module<sup>1</sup>, is replaced by a  $\mathbb{Z}[\lambda]$ -module  $\mathcal{I}_n$ . The algebraic integer  $\lambda$  replaces the multiplicative constant 2, and one is interested in the prime factorization of  $\lambda^n - 1$  in  $\mathbb{Z}[\lambda]$ , leading to ideal-theoretic considerations. The symbolic representation of hyperbolic toral automorphisms was considered in [15], and is related to the codings corresponding to expansions to non-integral bases, the so-called *beta-expansions* [2, 3, 29].

In this paper we consider arithmetic coding in a different context, namely the dynamics of renormalization of interval-exchange transformations (IET). The restriction of IETs to algebraic number fields has been the subject of recent investigations [20, 25]. It has been shown [25, theorem 1] that any uniquely ergodic renormalizable IET has the property that the lengths of its intervals are algebraic numbers. An algebraic number field  $K$  is thus associated to any such dynamical system. Furthermore, the renormalization constant is a unit in the ring of integers of  $K$ , and the interval lengths generate a module of maximal rank in  $K$ . Considering the restriction of these IETs to their field  $K$  of definition is then quite natural, and it leads to dynamics on lattices.

This paper is devoted to the analysis of the renormalization dynamics of a specific IET, proposed by Arnoux and Yoccoz [4], to be defined in section 2. Our work extends the investigation [20], which dealt with the lattice dynamics of the same model. This map features exact scaling, meaning that the first-return map on one of the intervals is a scaled version of the original map. The interval lengths are then algebraic numbers, that belong to a cubic field  $K$ . Its scaling dynamics—a subshift of finite type—leads to an arithmetic coding where the digits are algebraic integers in  $K$  rather than integers (a phenomenon already studied in unrelated contexts [12, 15]). The eventually periodic renormalization

---

<sup>1</sup> given a ring  $R$ , an  $R$ -module is an additive group equipped with multiplication by elements of  $R$ —see [13]

sequences coincide with the elements of the underlying number field.

We exploit the existence of a symbol space which can be used to represent both the dynamics of the map and the dynamics of scaling. The former has zero topological entropy, and it could be described as a generalized odometer; by contrast, the latter has positive entropy. This type of coding originates from the so-called *recursive tiling property* of a dynamical system: a scale-invariant set of tiles exists, which cover the entire space under iteration, in a hierarchical fashion. The symbol space consists of pairs of integers, which label a tile together with the tile's transit time measured from a reference position. These tilings have been used extensively in the study of piecewise isometric systems in two dimensions [16, 18, 19]. It turns out that these symbol sequences possess a rich arithmetical structure, which, to our knowledge, is largely unexplored. Once again, modular multiplication (by the scaling constant  $\omega$ ) will determine many properties of the renormalization dynamics. In this respect, the Arnoux-Yoccoz model is an ideal object of study; it illustrates how basic constructs developed elsewhere generalize, while remaining sufficiently simple for the numerical exploration of its asymptotics.

The system under study is defined in section 2, where some key results in [20] are reviewed. In section 3 we show that the periodic points of given period belong to the cartesian product of a finite module over an algebraic number ring—which depends on the period—and a finite set—the *core region*, which is period-independent. These two components represent the ‘fractional and integer parts’ of the periodic points, respectively. The finiteness of the core region derives from the Pisot property of the scaling factor. In section 4, we order the periodic points by increasing period, and consider asymptotic properties of their integer and fractional parts. We prove that the set of periodic points which share the fractional part with some other periodic point is infinite (theorem 3). Numerical evidence, however, strongly suggests that this set has zero density, which is expressed as conjecture 1. By embedding the fractional parts of the cycles into the unit cube, we study the relation between fractional and integer parts from a geometrical viewpoint. We collect some evidence that the phase space is organized in a highly nontrivial manner, which suggests directions for future investigations.

In section 5, we order the periodic points by increasing ‘denominator’, and relate their period to the ideal factorization of the denominator in the ring of integers of the underlying number field  $K$  (theorem 4). In this way a dynamical phenomenon—the number of renormalizations needed to map an orbit into itself—is linked to the arithmetical properties of the points in the orbit. Our experiments here investigate the asymptotic distribution of periods for large denominator, obtaining a picture consistent with that of increasing period, which provides additional support for our conjecture 1.

This research was supported by EPSRC grant No GR/S62802/01.

## 2 Preliminaries

Arnoux and Yoccoz [4], in their study of pseudo-Anosov diffeomorphisms, introduced a family of interval-exchange transformations defined over algebraic number fields of arbitrarily large degree, with the property of being renormalizable, namely of generating only finitely many induced maps, up to scaling. The simplest nontrivial member of this family corresponds to a cubic field; the object of our study is the scale-invariant (exactly renormalizable) mapping  $\rho$  induced by it on a sub-interval. This mapping  $\rho : [0, 1) \rightarrow [0, 1)$  is defined as follows [4, 20]

$$\rho(x) = x + \tau_j, \quad x \in \Omega_j \quad \Omega_j = [\delta_j, \delta_{j+1}) \quad j = 0, \dots, N-1 \quad (2)$$

where  $N = 7$ , the discontinuity points  $\delta_j$  (with  $\delta_N = 1$ ) and translations  $\tau_j$  are listed in table 1, and  $\lambda$  is the real root of the irreducible cubic polynomial

$$f(x) = x^3 + x^2 + x - 1. \quad (3)$$

| $j$ | $\delta_j$                       | $\tau_j$                    | $\nu_j$ | $p(j, t)$                               |
|-----|----------------------------------|-----------------------------|---------|-----------------------------------------|
| 0   | 0                                | $\lambda + \lambda^2$       | 4       | (0, 6, 3, 6)                            |
| 1   | $1 - \lambda - \lambda^2$        | $-1 + 3\lambda$             | 13      | (0, 6, 3, 6, 1, 6, 2, 5, 6, 1, 6, 3, 6) |
| 2   | $1 - 2\lambda + \lambda^2$       | $\lambda - \lambda^2$       | 12      | (0, 6, 3, 6, 1, 6, 2, 5, 6, 2, 4, 6)    |
| 3   | $(3 - 4\lambda - \lambda^2)/2$   | $-1 + 2\lambda + \lambda^2$ | 8       | (0, 6, 3, 6, 1, 6, 3, 6)                |
| 4   | $(1 - \lambda^2)/2$              | $1 - \lambda - \lambda^2$   | 8       | (0, 6, 4, 5, 6, 2, 4, 6)                |
| 5   | $(-1 + 2\lambda + 3\lambda^2)/2$ | $\lambda - \lambda^2$       | 12      | (0, 6, 4, 5, 6, 2, 5, 6, 1, 6, 3, 6)    |
| 6   | $\lambda$                        | $-\lambda$                  | 4       | (0, 6, 4, 6)                            |

Table 1: The data defining the cubic scale-invariant Arnoux-Yoccoz map  $\rho$ .

The map  $\rho$  corresponds to the permutation  $(0, 1, 2, 3, 4, 5, 6) \mapsto (6, 2, 4, 3, 5, 1, 0)$  of the  $N$  disjoint intervals  $\Omega_j$ , which constitute a partition of  $[0, 1)$ . The resulting dynamical system has zero topological entropy, being a piecewise isometry [8]; in addition, it is uniquely ergodic, and hence, in particular, it has no periodic orbits.

The map  $\rho$  is defined over the cubic number field

$$K = \mathbb{Q}(\lambda) = \{r_0 + r_1\lambda + r_2\lambda^2 \mid r_i \in \mathbb{Q}\}. \quad (4)$$

If we restrict the coefficients  $r_i$  in (4) to integer values, we obtain the ring  $\mathbb{Z}[\lambda]$  of all algebraic integers in  $\mathbb{Q}(\lambda)$ ; its group of units (invertible elements) has rank one, and is generated by  $\lambda$  [10, p. 519]. The unit  $\omega = \lambda^3$  is important: in fact, scaling by  $\omega$  conjugates the map  $\rho$  to the induced map on the leftmost interval  $\Omega_0 = [0, 1 - \lambda - \lambda^2) = [0, \omega)$ , which can be verified by direct calculation.

To the above data we associate the sequence space  $\mathcal{V}$  constituted by the sequences

$$\sigma = ((j_1, t_1), (j_2, t_2), \dots) \quad 0 \leq j_k < N, \quad 0 \leq t_k < \nu_k$$

where the  $j_k$  are subject to the constraints

$$j_k = p(j_{k+1}, t_{k+1}), \quad k \geq 1 \quad (5)$$

and  $p$  is the *path function* defined in table 1 (the sequence on the  $j$ th row of the table represents the values  $p(j, t), t = 0, \dots, \nu_j - 1$ ). Each sequence  $\sigma \in \mathcal{V}$  satisfying (5) defines a real number  $x$  via the equations

$$x(\sigma) = \sum_{i=1}^{\infty} d_i \omega^{i-1} \quad d_i = \sum_{t=0}^{t_i-1} \tau_{p(j_i, t)} \quad (6)$$

where the ‘digit’  $d_i = d_i(j_i, t_i)$  belongs to a *finite* subset of the  $\mathbb{Z}$ -module  $\mathcal{M}$  generated by the translations  $\tau_k$ . It is easy to see that  $\mathcal{M} = \mathbb{Z}[\lambda]$ ; furthermore, there are  $\sum \nu_j = 61$  valid symbols, corresponding to 25 distinct values of the digits.

By construction, the point  $x(\sigma)$  belongs to the unit interval. The set of points sharing a finite digit sequence  $(d_1, \dots, d_n)$  is a half-open interval, being the intersection of translated and scaled versions of the original intervals  $\Omega_j$ . We call it a *tile* of level  $n$ . For each integer  $n$ , the tiles of level  $n$  constitute a partition of the unit interval. This is the *recursive tiling property* of the mapping, which can be verified to hold for  $n = 1$ , and then to follow from scaling. The representation (6) may be regarded as a variant of the expansion of real numbers in non-integral bases—the so called  $\beta$ -expansions. Our basis  $\omega$  is the reciprocal of a Pisot number<sup>2</sup>, a property which is of importance here (it underpins theorems 1, 2 below) as it is in the theory of  $\beta$ -expansions [1, 29]. We emphasize again that the digits of our  $\omega$ -expansion are algebraic integers.

The recursive tiling property implies the surjectivity of the map  $\sigma \mapsto x(\sigma)$ . Such a map is not injective, but the lack of injectivity is easily controlled. Whenever it fails, there are precisely two codes corresponding to the same point, and these codes are eventually periodic and entirely characterized by their periodic part. This degeneracy can be removed by stipulating that the nested set of tiles specified by the code  $\sigma$  must also contain its limit point  $x$ . If only such codes are allowed, the correspondence between points and codes is bijective. This amounts to excluding the following six periodic tails

$$(1, 9)^\infty \quad (2, 6)^\infty \quad (3, 2)^\infty \quad (4, 6)^\infty \quad (5, 3)^\infty \quad (6, 1)^\infty. \quad (7)$$

(The situation is analogous to the decimal representation of rational numbers: the tail  $(9)^\infty$  is excluded to achieve a unique representation.)

With the above stipulation on valid codes, the map

$$\gamma : [0, 1) \rightarrow [0, 1) \quad x \mapsto (x - d_1)\omega^{-1} \quad (8)$$

is conjugate to the left shift on  $\mathcal{V}$ . This is a subshift of finite type, which we call the *scaling dynamics* of the Arnoux-Yoccoz map. The transitivity of the incidence matrix corresponding to the admissibility conditions (5) can be verified directly. It then follows [14, p. 51], that  $\gamma$  is mixing, and that the periodic orbits are dense.

The following result was proved in [20].

---

<sup>2</sup>A Pisot number is an algebraic number greater than one, whose algebraic conjugates other than itself lie inside the complex unit circle.

**Theorem 1** *A point  $x$  of the unit interval belongs to  $\mathbb{Q}(\lambda)$  if and only if the code  $\sigma(x)$  is eventually periodic.*

This result calls to mind the familiar property of expansions in integral bases, where the set of points characterized by the eventual periodicity of the digits is the field  $\mathbb{Q}$ . To place this theorem into context, we note the general results of Bertrand [6] and Schmidt [28] on eventually periodic greedy expansions in a Pisot base.

To study dynamics over  $\mathbb{Q}(\lambda)$  we consider the decomposition

$$\mathbb{Q}(\lambda) = \Xi + \mathbb{Z}[\lambda]$$

where

$$\Xi = \{\xi_0 + \xi_1\lambda + \xi_2\lambda^2 \mid \xi_i \in \mathbb{Q} \cap [0, 1)\}.$$

This prescription gives a unique representation of  $x \in \mathbb{Q}(\lambda)$  as  $x = \xi + \beta$ , with  $\xi \in \Xi$  and  $\beta \in \mathbb{Z}[\lambda]$ . Accordingly, we define the projections to the two components

$$\Pi_f : \mathbb{Q}(\lambda) \rightarrow \Xi \quad x \mapsto \xi, \quad \Pi_i : \mathbb{Q}(\lambda) \rightarrow \mathcal{B} \quad x \mapsto \beta. \quad (9)$$

These components should be viewed, respectively, as the ‘fractional and integer parts’ of the field elements. Note that  $\Xi$  is a  $\mathbb{Z}[\lambda]$ -module, isomorphic to  $\mathbb{Q}(\lambda)/\mathbb{Z}[\lambda]$ ; this fact will be of importance later.

In the rest of this paper we shall be concerned with the set of periodic points of  $\gamma$ , a  $\gamma$ -invariant set which belongs to  $\mathbb{Q}(\lambda) \cap [0, 1)$ .

### 3 Periodic points of scaling dynamics

Consider a point  $x$  with strictly periodic code

$$\sigma(x) = ((j_1, t_1), \dots, (j_n, t_n))^\infty.$$

From theorem 1 we have that  $x \in \mathbb{Q}(\lambda) \cap [0, 1)$ , and a straightforward calculation shows that

$$x = \frac{1}{1 - \omega^n} \sum_{i=1}^n d_i \omega^{i-1} \quad (10)$$

where the digits  $d_i$  were defined in (6). The following result —proved in [20]— is crucial to our investigation

**Theorem 2** *There exists a finite subset  $\mathcal{B}$  of  $\mathbb{Z}[\lambda]$  such that any  $\gamma$ -periodic point  $x$  can be represented as*

$$x = \xi + \beta \quad \xi \in \Xi, \beta \in \mathcal{B}. \quad (11)$$

*Conversely, for every  $\xi \in \Xi$ , there exists  $\beta \in \mathcal{B}$  such that  $x = \xi + \beta$  is periodic under  $\gamma$ .*

The set  $\mathcal{B} \in \mathbb{Z}[\lambda]$  is called the *core region*. The theorem says that the points which are periodic under the scaling map  $\gamma$  can have only finitely many distinct integer parts. Exact computations give the estimates

$$31 \leq \#\mathcal{B} \leq 348. \quad (12)$$

The upper bound was established using a technique developed in [20]. (In that reference, an upper bound of 225 was obtained for the projection of the core region to a two-dimensional lattice.) The lower bound will be established below —see table 3.

Theorem 2 outlines the basic structure of the periodic points of  $\gamma$ , yet important questions concerning their integer and fractional parts are not addressed. Are there infinitely many fractional parts  $\xi \in \Xi$  shared by more than one periodic point? Is it true that for all  $\beta$  in the core region there are infinitely many periodic points with integer part equal to  $\beta$ ? If so, do their fractional parts form a set of positive density?

To investigate these issues, for  $\beta \in \mathcal{B}$  we consider the set  $\Xi(\beta)$  of all  $\xi \in \Xi$  for which  $\xi + \beta$  is periodic. Using the projections defined in (9), we see that

$$\Xi(\beta) = \Pi_f \circ \Pi_\beta^{-1}(\beta) \quad (13)$$

and from theorem 2 we know that

$$\bigcup_{\beta \in \mathcal{B}} \Xi(\beta) = \Xi. \quad (14)$$

Next, for  $\xi \in \Xi$  we define the *multiplicity*  $\kappa(\xi)$  to be the number of  $\gamma$ -periodic points with fractional part equal to  $\xi$ . From theorem 2 we know that  $1 \leq \kappa(\xi) \leq \#\mathcal{B}$ , and we are interested in the set  $\partial\Xi$  of those  $\xi$  that correspond to more than one periodic point. This set is given by

$$\partial\Xi = \{\xi \in \Xi : \kappa(\xi) > 1\} = \bigcup_{\substack{\beta, \beta' \in \mathcal{B} \\ \beta \neq \beta'}} (\Xi(\beta) \cap \Xi(\beta')). \quad (15)$$

For reason that will become clear later, we call  $\partial\Xi$  the *boundary set*.

We begin our study of periodic points by seeking an analogue of equation (1). Let  $M_n$  be the smallest positive integer which is divisible by  $1 - \omega^n$  in  $\mathbb{Z}[\lambda]$ , that is,

$$(1 - \omega^n)\alpha_n = M_n \quad (16)$$

for  $M_n$  minimal and some  $\alpha_n \in \mathbb{Z}[\lambda]$ . From equation (10) one sees that  $M_n x \in \mathbb{Z}[\lambda]$ , that is,

$$\text{Fix}(\gamma^n) \subset \frac{1}{M_n} \mathbb{Z}[\lambda].$$

Equivalently,  $M_n$  is the least common multiple of the denominators of all  $n$ -periodic points. In practise,  $M_n$  is computed efficiently with the extended Euclid's algorithm.

By construction,  $M_n$  is a divisor of the norm  $N(1 - \omega^n)$  of  $1 - \omega^n$  (the norm of an algebraic number is the product of its algebraic conjugates). While the value of  $N(1 -$

$\omega^n$ ), is well-behaved —see section 4, equations (20,21)— the value of  $M_n$  features large fluctuations. In this respect, we note the factorization

$$1 - \omega^n = - \prod_{d|n} C_{3d}(\lambda) \prod_{\substack{d|n \\ 3 \nmid d}} C_d(\lambda) \quad (17)$$

where  $C_d(x)$  is the  $d$ -th cyclotomic polynomial [23, p 37]. This formula is established starting from the polynomial identity  $x^n - 1 = \prod_{d|n} C_d(x)$ , recalling that  $\omega = \lambda^3$ , and then relating the divisors of  $3n$  to those of  $n$ . The formula provides only a partial factorization of  $1 - \omega^n$ , because the irreducibility of the cyclotomic polynomial  $C_d(x)$  in  $\mathbb{Z}[x]$  does not imply that  $C_d(\lambda)$  are primes in  $\mathbb{Z}[\lambda]$  (which is a principal ideal domain [10, p. 519]).

For example, the identities

$$\omega - 1 = C_1(\lambda)C_3(\lambda) = (\lambda - 1)\lambda^{-1} \quad (1 - \lambda)^3\lambda^{-5} = 2$$

together with the fact that  $\lambda$  is a unit, show that the denominator of a periodic point always has the irreducible factor  $1 - \lambda$ , which is a prime divisor of 2. From the above observation, it follows that  $M_n$  is even. Likewise,  $C_6(\lambda)$  is a divisor of 7, and so if  $n$  is even,  $M_n$  is divisible by 14, etc.

The periodic orbits of the scaling dynamics can be ordered in two ways, namely by increasing period  $n$ , or by increasing denominator  $m$  (the choice of sub-orderings for given  $n$  or  $m$  are not important here). As mentioned above, these orderings lead to different perspectives (see also [9, section 6]), and they will be considered in sections 4 and 5, respectively. In analogy with the rational case, whenever  $1 - \omega^n$  has a large number of divisors, cancellation becomes likely in equation (10), leading to  $n$ -cycles with denominator much smaller than  $M_n$ . As the denominator is increased, these are the cycles that are found first.

Each ordering gives asymptotic information about the sets  $\Xi(\beta)$  and  $\partial\Xi$  defined above. This will be our main object of study.

## 4 Ordering by period

Let the period  $n$  be fixed. With reference to equations (9) and (11), we define

$$\mathcal{I}_n = \Pi_f \left( \frac{1}{1 - \omega^n} \mathbb{Z}[\lambda] \right) \quad \mathcal{I}'_n = \Pi_f (\text{Fix}(\gamma^n)) \quad \mathcal{B}_n = \Pi_i (\text{Fix}(\gamma^n)). \quad (18)$$

From equation (10) it follows that the fractional part of the  $n$ -cycles is contained in  $\mathcal{I}_n$ , and hence  $\mathcal{I}'_n \subset \mathcal{I}_n$ . By construction, the fractional parts in the (possibly empty) residual set  $\mathcal{I}_n \setminus \mathcal{I}'_n$  belong to cycles whose period is a multiple of  $n$ .

To determine the cardinality of  $\mathcal{I}_n$ , we must refine the decomposition (11). Let  $m$  be a positive integer, and let  $\Xi_m$  be the set of  $\xi$ -points with denominator  $m$  namely

$$\Xi_m = \Pi_f \left( \frac{1}{m} \mathbb{Z}[\lambda] \right) \quad (19)$$

which is again a  $\mathbb{Z}[\lambda]$ -module, and is isomorphic (via the map  $\varphi : x \mapsto mx$ ) to the module  $\mathbb{Z}[\lambda]/m\mathbb{Z}[\lambda]$ , which has  $m^3$  elements.

Consider now the algebraic integer  $\alpha_n$  defined in equation (16). The map  $\varphi$  sends  $\mathcal{I}_n$  to the ideal generated by  $\alpha_n$  in  $\mathbb{Z}[\lambda]/M_n\mathbb{Z}[\lambda]$  (now regarded as a ring), which yields the equation

$$\#\Xi_{M_n} = \#\mathcal{I}_n \times |N(\alpha_n)|.$$

Using the multiplicativity of the norm, and the fact that  $\#\Xi_M = N(M) = M^3$ , we obtain the formula

$$\#\mathcal{I}_n = |N(1 - \omega^n)| \quad (20)$$

which, together with equation (16), shows that  $\#\mathcal{I}_n$  is a multiple of  $M_n$ . Explicit computation gives

$$\#\mathcal{I}_n = \omega^{-n} - \omega^n - 2 \left( \sqrt{\omega^{-n}} - \sqrt{\omega^n} \right) \cos(n\theta) \quad (21)$$

where

$$\cos(\theta) = \frac{1}{2}\sqrt{\omega}(5 - \omega).$$

To compute  $\#\text{Fix}(\gamma^n)$  we consider the incidence matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 & 1 & 6 \\ 1 & 1 & 2 & 1 & 1 & 1 & 5 \\ 1 & 1 & 0 & 2 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 & 2 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 & 5 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

where  $A_{i,j}$  is the number of times that the first-return orbit of the scaled interval  $\omega\Omega_i$  visits the interval  $\Omega_j$ , obtained from the path function data of table 1. The characteristic polynomial of  $A$  factors into irreducibles as  $(x-1)(x^3-5x^2+7x-1)(x^3-7x^2+5x-1)$ , whose real roots are 1,  $\omega = \lambda^3$ , and  $\omega^{-1}$ , respectively. Accounting for the six forbidden period-1 codes listed in (7), we find

$$\#\text{Fix}(\gamma^n) = \text{Tr}A^n - 6 = \omega^{-n} + \omega^n + 2(\sqrt{\omega^{-n}} + \sqrt{\omega^n}) \cos(n\theta) - 5. \quad (22)$$

Expressions (21) and (22) have the same leading term  $\omega^{-n}$  (recall that  $|\omega| < 1$ ), and hence  $\#\text{Fix}(\gamma^n) \sim \#\mathcal{I}_n$ . Furthermore

$$\#\text{Fix}(\gamma^n) - \#\mathcal{I}_n = 4\sqrt{\omega^{-n}} \cos(n\theta) + 2\omega^n - 5 \sim 4\sqrt{\omega^{-n}} \cos(n\theta). \quad (23)$$

This formula shows that for some values of  $n$ ,  $\#\text{Fix}(\gamma^n)$  is greater than  $\#\mathcal{I}_n$ , which implies that there exist distinct periodic points with the same fractional part  $\xi$ . In fact, the asymptotic expression (23) shows that this must happen infinitely often, and that the population of points with multiplicity greater than 1 is maximal when the period is a denominator of the continued fraction expansions of  $\theta/2\pi$ . Recalling the definition (15) of the boundary set, we have the following result.

**Theorem 3** *The boundary set  $\partial\Xi$  is infinite.*

Using induction, it is possible to prove that the periodic codes

$$((1, 4)^k, (3, 4), (1, 2))^\infty \quad ((2, 9)^k, (4, 5), (2, 10))^\infty \quad k = 0, 1, \dots$$

(for  $k = 0$  the symbol is deleted) correspond to points of period  $k + 2$  on the boundary set. Now, the fixed points  $(1, 4)^\infty$  and  $(2, 9)^\infty$  also belong to  $\partial\Xi$ , as easily verified. This shows that the boundary set contains points of any period, which is stronger than theorem 3. We shall not produce this proof here.

When  $\#\text{Fix}(\gamma^n) < \#\mathcal{I}_n$ , the  $\xi$ -values do not exhaust the whole of  $\mathcal{I}_n$ , and formula (23) again shows that this must happen infinitely often. Since  $\#\mathcal{I}'_n \# \mathcal{B} \geq \#\text{Fix}(\gamma^n)$ , we obtain

$$\frac{\#\mathcal{I}'_n}{\#\mathcal{I}_n} \geq \frac{\#\text{Fix}(\gamma^n)}{\#\mathcal{I}_n \# \mathcal{B}} \rightarrow \frac{1}{\# \mathcal{B}} \quad \text{as } n \rightarrow \infty. \quad (24)$$

This bound is far from optimal. To see this, we define

$$\mathcal{I}''_n = \{\xi \in \mathcal{I}'_n : \kappa(\xi) > 1\}.$$

Constructing explicitly all  $n$ -periodic points for  $n \leq 14$ , gives the data displayed in table 2.

| $n$ | $\#\mathcal{I}''_n$ | $\#\mathcal{I}''_n / \#\text{Fix}(\gamma^n)$ | $\#\mathcal{B}_n$ |
|-----|---------------------|----------------------------------------------|-------------------|
| 1   | 2                   | 0.2857                                       | 7                 |
| 2   | 14                  | 0.3111                                       | 17                |
| 3   | 38                  | 0.1467                                       | 21                |
| 4   | 86                  | 0.05595                                      | 22                |
| 5   | 182                 | 0.01939                                      | 24                |
| 6   | 374                 | 0.006440                                     | 24                |
| 7   | 758                 | 0.002100                                     | 27                |
| 8   | 1534                | 0.0006831                                    | 30                |
| 9   | 3170                | 0.0002268                                    | 30                |
| 10  | —                   | —                                            | 30                |
| 11  | —                   | —                                            | 30                |
| 12  | —                   | —                                            | 30                |
| 13  | —                   | —                                            | 30                |
| 14  | —                   | —                                            | 31                |

Table 2: Multiplicity and core region data from orbits of period  $n$ .

The multiplicity data are limited to the range  $n \leq 9$ , because the comparison of all fractional parts requires that these quantities be stored. The most significant finding, to be used in conjunction with the asymptotic formulae (21) and (22), is that distinct periodic points typically have distinct fractional parts; namely, the set of cycles with multiplicity 1 appears to have full density. Furthermore, the ratio  $\#\mathcal{I}''_n / \#\text{Fix}(\gamma^n)$  appears to be tending rather rapidly to zero, with a scaling ratio close to  $1/3$ . In the computations to be described in section 5 we observe the same phenomenon when the cycles are ordered by increasing denominator, leading to the following conjecture.

**Conjecture 1** *The following holds*

$$\lim_{n \rightarrow \infty} \frac{\#\{\xi \in \mathcal{I}'_n : \kappa(\xi) = 1\}}{\#\mathcal{I}_n} = \lim_{m \rightarrow \infty} \frac{\#\{\xi \in \Xi_m : \kappa(\xi) = 1\}}{m^3} = 1.$$

The validity of this conjecture would imply that, as  $n \rightarrow \infty$

$$\#\mathcal{I}'_n \sim \#\mathcal{I}_n \sim \omega^{-n}$$

to be compared with the bound (24).

Regarding the core region data of table 2, the lower bound of 31 is substantially smaller than the upper bound of 348 given in (12). With reference to equation (11), we display in table 3 some elements  $\beta = m_0 + m_1\lambda + m_2\lambda^2$  of the core region, represented as integer triples  $(m_0, m_1, m_2)$ . Each value of  $\beta$  has a probability  $\mu(\beta)$ , defined as the limiting density of the  $\xi$ -values that correspond to it. Approximate values of these probabilities may be computed in two different ways, corresponding to ordering by period (the second and third columns in the table) and by denominator (the fourth column). In the former case, the data was computed from the 14-cycles (130399019341 data points), which project to the 31 points of the core region listed in the first column, respectively. The data for period 13 are also shown, to give an idea of convergence. The probability  $\mu'$  computed using all cycles with ‘denominator’  $m \leq 200$  is displayed in the third column (373112717 data points, which project to a subset of 28 core region points) —see the next section. While it is conceivable that the core region could contain more points than those displayed in table 3, the stability of the above figures suggests that they are a reliable approximation to the densities.

So far we have expressed probabilistic information in terms of densities; we close this section by discussing related questions of measure. To this end, we consider the embedding of the fractional parts into the unit cube

$$\varphi : \Xi \rightarrow [0, 1)^3 \quad r_0 + r_1\lambda + r_2\lambda^2 \mapsto (r_0, r_1, r_2), \quad 0 \leq r_i < 1. \quad (25)$$

This construct depends on the particular basis chosen for  $\mathbb{Q}(\lambda)$ . However, a change of basis corresponds to a unimodular transformation, which, if we identify the unit cube with the 3-torus, is continuous and volume preserving. So any property of the (closure of) embedded objects which is topological or concerns three-dimensional Lebesgue measure will be independent of the basis.

It is natural to consider sequences of points that are uniformly distributed in the cube; for instance, the sequence of fractional parts of periodic points with increasing period, or with increasing denominator. Consider now the embedding  $\varphi(\partial\Xi)$  of the boundary set  $\partial\Xi$ , defined in equation (15). Recalling that  $\#\text{Fix}(\gamma^n) \sim \omega^{-n} = \lambda^{-3n}$  (see equation (22)), the scaling law observed in table 2 is consistent with the closure of  $\varphi(\partial\Xi)$  having zero three-dimensional Lebesgue measure. Now, the points of the boundary set are precisely the points with multiplicity greater than one; so any uniformly distributed sequence will have the property that the density of the elements with multiplicity greater than 1 is zero. Thus the vanishing of the measure of the boundary set in the cube implies conjecture 1.

We have examined the arrangements of the boundary set in the cube, which gives some support to this conjecture, although the available data are not quite conclusive. In fact, there is an even stronger property that should be considered, namely that the core region localizes in the unit cube; this means that the measure of the closure of the sets  $\varphi(\Xi(\beta))$

| $\beta$      | $\mu_{13}$             | $\mu_{14}$              | $\mu'$                 |
|--------------|------------------------|-------------------------|------------------------|
| ( 0, 0, 0)   | 0.21275                | 0.21275                 | 0.21566                |
| ( 0, -1, -1) | 0.18921                | 0.18921                 | 0.18811                |
| ( 0, 0, -1)  | 0.13777                | 0.13777                 | 0.13688                |
| ( 1, -2, -1) | 0.095625               | 0.095625                | 0.096146               |
| (-1, 0, 0)   | 0.077156               | 0.077155                | 0.075485               |
| (-1, 0, 1)   | 0.062212               | 0.062213                | 0.062157               |
| ( 0, -2, 0)  | 0.57953                | 0.057952                | 0.057331               |
| ( 0, -1, 1)  | 0.029309               | 0.029308                | 0.029370               |
| ( 1, -3, 0)  | 0.026513               | 0.026514                | 0.026326               |
| ( 1, -2, 0)  | 0.026416               | 0.026416                | 0.027329               |
| ( 0, -2, -1) | 0.022015               | 0.022014                | 0.020936               |
| ( 0, -2, 1)  | 0.013304               | 0.013304                | 0.013345               |
| ( 0, -1, 0)  | 0.011008               | 0.011009                | 0.010908               |
| ( 0, 0, 1)   | 0.0088052              | 0.0088053               | 0.0093162              |
| (-1, 1, 0)   | 0.0079820              | 0.0079830               | 0.0083504              |
| (-1, 1, 1)   | 0.0068145              | 0.0068150               | 0.0069532              |
| ( 0, 1, 0)   | 0.0056184              | 0.0056188               | 0.0058062              |
| ( 1, -1, -1) | 0.0052609              | 0.0052609               | 0.0056897              |
| ( 1, -3, -1) | 0.0020413              | 0.0020418               | 0.0019191              |
| (-1, 0, -1)  | 0.0011704              | 0.0011707               | 0.0010408              |
| (-1, -1, 1)  | 0.00075012             | 0.00075025              | 0.00065573             |
| ( 1, -4, 0)  | 0.00020058             | 0.00020069              | 0.00018132             |
| ( 0, -3, 0)  | $7.5098 \cdot 10^{-5}$ | $7.5092 \cdot 10^{-5}$  | $5.2630 \cdot 10^{-5}$ |
| ( 1, -2, 1)  | $2.1778 \cdot 10^{-5}$ | $2.1757 \cdot 10^{-5}$  | $3.2518 \cdot 10^{-5}$ |
| ( 1, -3, 1)  | $3.7165 \cdot 10^{-6}$ | $3.7439 \cdot 10^{-6}$  | $7.8207 \cdot 10^{-6}$ |
| ( 0, -1, -2) | $3.6249 \cdot 10^{-6}$ | $3.6509 \cdot 10^{-6}$  | $1.1444 \cdot 10^{-6}$ |
| ( 1, 0, -1)  | $3.3849 \cdot 10^{-6}$ | $3.4058 \cdot 10^{-6}$  | $7.8207 \cdot 10^{-6}$ |
| ( 1, -1, -2) | $2.6184 \cdot 10^{-7}$ | $2.6498 \cdot 10^{-7}$  | 0                      |
| ( 0, -3, 1)  | $2.6184 \cdot 10^{-7}$ | $2.6498 \cdot 10^{-7}$  | $4.3151 \cdot 10^{-7}$ |
| (-1, 1, -1)  | $9.1552 \cdot 10^{-8}$ | $9.3050 \cdot 10^{-8}$  | 0                      |
| (-1, -1, 0)  | 0                      | $1.2873 \cdot 10^{-11}$ | 0                      |

Table 3: The elements  $\beta$  of the core region  $\mathcal{B}$ , with three distinct estimates  $\mu_{13}$ ,  $\mu_{14}$  and  $\mu'$  of the associated densities. The first two were computed using all cycles up to periods 13 and 14, respectively, and the third using all cycles with denominator not exceeding 200. The floating-point numbers represent exact rationals rounded off to 5-digit precision.

add up to unity, and represent the probabilities  $\mu(\beta)$  estimated in table 3. (Localization techniques have proved very useful in the study of certain lattice maps [7, 17, 30].)

These problems deserve further investigation.

## 5 Ordering by denominator

Let  $m$  be a positive integer. We consider the periodic orbits of the map  $\gamma$  having denominator  $m$ . According to theorem 2 and equation (19), these points belong to the set  $\Xi_m + \mathcal{B}$ . We are interested in determining their period.

In section 4 we have seen that scaling by  $m$  maps  $\Xi_m$  into the ring  $\mathbb{Z}[\lambda]/m\mathbb{Z}[\lambda]$ , which for our purpose is a more convenient representation. If  $\alpha, \beta \in \mathbb{Z}[\lambda]$ , we denote by  $(\alpha)$ ,  $(\alpha, \beta)$ , etc., the ideals they generate. Thus  $(1) = \mathbb{Z}[\lambda]$ .

Let  $\xi$  be a point in  $\mathbb{Z}[\lambda]/(m)$ . The *order*  $t(\xi)$  of  $\xi$  is defined as

$$t(\xi) = \min\{k \geq 1 : \omega^k \xi \equiv \xi \pmod{(m)}\}. \quad (26)$$

From theorem 2, we have that every  $\xi \in \Xi$  is the fractional part of at least one periodic point; it follows that the order of  $\xi$  must divide the corresponding period. In fact, from conjecture 1 it would follow that, asymptotically, the period of a periodic point is almost surely equal to the order of its fractional part. Now  $\#\Xi_m = m^3$ , which gives the crude bound  $t(\xi) \leq m^3$ .

To obtain a sharper bound for  $t$ , we must consider the ideal factorization of  $m$  in  $\mathbb{Z}[\lambda]$ . We describe it in terms of the factorization of  $f(x)$  modulo a prime  $p$ . Since  $\mathbb{Z}[\lambda]$  is the ring of all algebraic integers in  $\mathbb{Q}(\lambda)$ , the discriminant of  $f(x)$ , which is equal to  $-44$ , does not have any spurious prime divisor, so that the factorization of  $f(x)$  modulo  $p$  describes the ideal factorization of  $p$ , without exceptions [21, theorem 27]. The primes that divide the discriminant are  $p = 2, 11$ ; they *ramify*, that is,  $f(x)$  has multiple roots modulo  $p$ . For all the other primes  $p$ , the polynomial  $f(x) \pmod{p}$  has distinct factors. A prime  $p$  is *inert*, *splits*, or *splits completely*, respectively, if  $f(x)$  decomposes into the product of 1, 2, or 3 irreducible factors, respectively. Each possibility occurs infinitely often, and indeed with probability  $1/3$ ,  $1/2$ , or  $1/6$ , respectively. This follows from Chebotarev's density theorem and the fact that the Galois group of  $f(x)$  is  $S_3$  [26, p. 129].

The following result relates the order of a point  $\xi$  to the ideal factorization of its denominator.

**Theorem 4** *Let  $\xi \in \mathbb{Z}[\lambda]$ , and let  $m$  be a positive integer. Then  $t(\xi)$  is a divisor of  $T$ , where  $T = T(m)$  is computed as follows:*

- (i) *If  $\gcd(n, m) = 1$ , then  $T(nm) = \text{lcm}(T(n), T(m))$ .*

(ii) If  $m = p^e$  with  $p$  prime and  $e \geq 1$ , we have

| $p$              | $T(p^e)$                                                                                                                                        |
|------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|
| <i>ramif.</i>    | $\begin{cases} p^{e+1}(p-1) & \text{if } p = 2 \\ p^e(p-1) & \text{if } p = 11 \end{cases}$                                                     |
| <i>inert</i>     | $\begin{cases} p^{e-1}(p^2 + p + 1) & \text{if } p \equiv 0, 2 \pmod{3} \\ p^{e-1}(p^2 + p + 1)/3 & \text{if } p \equiv 1 \pmod{3} \end{cases}$ |
| <i>splits</i>    | $p^{e-1}(p^2 - 1)/3$                                                                                                                            |
| <i>splits c.</i> | $\begin{cases} p^{e-1}(p-1) & \text{if } p \equiv 2 \pmod{3} \\ p^{e-1}(p-1)/3 & \text{if } p \equiv 1 \pmod{3}. \end{cases}$                   |

**Proof.** With reference to the the congruence (26), we see that if  $\xi$  and  $m$  are coprime (meaning that the ideal identity  $(\xi, m) = (1)$  holds), then upon division by  $\xi$ , we obtain  $\omega^t \equiv 1 \pmod{(m)}$ , independent of  $\xi$ , which shows that  $t(\xi)$  is the multiplicative order of  $\omega$  in the ring  $\mathbb{Z}[\lambda]/(m)$ . Thus  $t(\xi)$  is a divisor of the order of the multiplicative group of this ring. From the decomposition

$$\frac{\mathbb{Z}[\lambda]}{(m)} \simeq \bigoplus_i \frac{\mathbb{Z}[\lambda]}{J_i}$$

where  $J_i$  are the pairwise coprime ideals divisors of  $(m)$ , we have, in particular, that  $T(m)$  can be computed as the least common multiple of its value at each summand.

If  $\xi$  and  $m$  are not coprime, then division by  $\xi$  leads to a congruence modulo the ideal  $(m)/(\xi, m)$ . The multiplicative group of the corresponding finite ring is a divisor of that considered above, so the same estimates apply.

Next we consider primary factors  $m = p^e$ , beginning with the case  $e = 1$ . We denote by  $\mathbb{F}_{p^k}$  the field with  $p^k$  elements, and by  $\mathbb{F}_{p^k}^*$  its multiplicative group. We shall use the fact that  $\omega = \lambda^3$  is a unit of norm 1, and that the prime  $p = 3$  is inert.

If  $p$  is inert, we have the ring isomorphism

$$\frac{\mathbb{Z}[\lambda]}{(p)} \simeq \mathbb{F}_{p^3}.$$

The Galois group of  $\mathbb{F}_{p^3}$  over  $\mathbb{F}_p$  is cyclic of order 3, and is generated by the Frobenius automorphism  $\mathcal{F} : \alpha \mapsto \alpha^p$ . The orbits of  $\mathcal{F}$  are the sets of algebraic conjugates. Denoting again by  $\omega$  its reduction to  $\mathbb{F}_{p^3}$ , we find

$$\omega^{1+p+p^2} = \mathcal{F}^0(\omega) \mathcal{F}^1(\omega) \mathcal{F}^2(\omega) = 1.$$

The last equality derives from the fact that the terms in the product are the algebraic conjugates of  $\omega$ , and since they are distinct, their product is the norm of  $\omega$  modulo  $p$ . So the order of  $\omega$  divides  $p^2 + p + 1$ . If  $p \equiv 1 \pmod{3}$ , then  $p^2 + p + 1 \equiv 0 \pmod{3}$ , and  $\omega$

has a cube root in  $\mathbb{F}_{p^3}$ , which is also a unit of norm 1. In this case the order of  $\omega$  divides  $(p^2 + p + 1)/3$ .

If  $p$  splits then

$$\frac{Z[\lambda]}{(p)} \simeq \mathbb{F}_p \oplus \mathbb{F}_{p^2}.$$

The order of  $\mathbb{F}_p^*$  divides that of  $\mathbb{F}_{p^2}^*$ , which is equal to  $p^2 - 1$ . Because  $p \neq 3$ , we have  $p^2 - 1 \equiv 0 \pmod{3}$ , hence  $\omega$  always has a cube root in  $\mathbb{F}_{p^2}$ .

If  $p$  splits completely, then

$$\frac{Z[\lambda]}{(p)} \simeq \mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p.$$

The order of the three reductions of  $\omega$  to each finite field is a divisor of the order of  $\mathbb{F}_p^*$ , which is  $p-1$ . If  $p \equiv 2 \pmod{3}$ , there are no further constraints on  $T(p)$ . If  $p \equiv 1 \pmod{3}$ , then  $\omega$  has a cube root in  $\mathbb{F}_p$ , and hence the subgroup it generates has index at least 3.

It remains to deal with the ramified primes 2 and 11. The prime 2 is totally ramified

$$f(x) \equiv (x + a)^3 \pmod{p} \quad p = 2, a = 1$$

leading to the ideal factorization  $(p) = P^3 = (p, \lambda + a)^3$ . The ideal  $P$  contains  $p^2$  incongruent points modulo  $(p)$ , and hence the multiplicative group of  $\mathbb{Z}[\lambda]/(p)$  has order  $p^3 - p^2 = p^2(p - 1)$ .

The prime 11 is partially ramified. We find

$$f(x) \equiv (x + a)^2(x + b) \pmod{p} \quad p = 11, a = 3, b = 6$$

which corresponds to the ideal factorization,  $(p) = P_1^2 P_2 = (p, \lambda + a)^2(p, \lambda + b)$ . Hence

$$\frac{Z[\lambda]}{(p)} \simeq R \oplus \mathbb{F}_p$$

where  $R$  is a ring with  $p^2$  elements,  $p$  of which are not invertible. Thus the multiplicative group of  $R$  has order  $p^2 - p = p(p - 1)$ , while that of  $\mathbb{F}_p$  has order  $p - 1$ , hence the result.

Finally, for  $e > 1$ , we apply a standard lifting argument. Let  $k$  be the largest integer for which  $T(p^k) = T(p)$ . (Such an integer  $k$  exists and is effectively computable.) That is,

$$\omega^{T(p)} = 1 + p^k \beta$$

where  $\beta \in \mathbb{Z}[\lambda]$  is such that  $(\beta, p) = (1)$ . Then, using the binomial theorem, we have

$$\omega^{T(p)p} = 1 + p^{k+1} \beta'$$

where again  $(\beta', p) = (1)$ . An easy induction on  $e$  shows that the order increases regularly by a factor  $p$  at each step. Thus

$$T(p^e) = T(p) p^{\max(0, e-k)}.$$

This completes the proof. □

In the totally ramified case, the  $\xi$ -dynamics is quite regular. From the fact that  $(2) = 2_1 = (2, \lambda + 1)^3$ , and  $\omega = 1 - \lambda - \lambda^2$ , we have  $\omega - 1 = -\lambda(\lambda + 1) \in 2_1$ . Thus, for any  $\xi \in \mathbb{Z}[\lambda]$ , we have that  $\omega\xi \equiv \xi \pmod{2_1}$ , and hence every orbit of  $\omega$  modulo  $(2^e)$  consists of congruent points modulo  $2_1$ . So, if the period is maximal, orbits are cosets of a lattice.

The theorem states that  $t(\xi)$  is a divisor of  $T$ , but gives no information as to the actual value of  $T/t$ . The exact value of  $t(\xi)$  is (essentially) determined by the order of the image of  $\omega$  in various finite fields, which cannot be computed in polynomial time. One would expect that the occurrence of each ratio  $T/t = 1, 2, \dots$  will be accompanied by large fluctuations and by a limiting probability. Establishing the existence of such a sequence of probabilities is a difficult problem, of the kind mentioned in the introduction in connection with Artin's conjecture.

Theorem 4 generalizes, with appropriate modifications, to any cubic field. From [25, theorem 1] we know that the scaling constant  $\omega$  is a unit: the fact that our  $\omega$  is the third power of a fundamental unit has merely brought about some specialization in the formulae, expressed via congruences. The presence of the split case in theorem 4 is a consequence of the fact that the Galois group of  $f(x)$  is the symmetric group. If the Galois group of  $f(x)$  is cyclic, the split case does not occur, while the other formulae in the theorem remain the same. In this case the Chebotarev's densities for the allowed factorizations become  $1/3$  for primes splitting completely, and  $2/3$  for inert primes.

We have constructed all periodic points with denominator not exceeding 200, obtaining 373112717 points in total. These points have the form  $x = \xi + \beta$  with  $\beta \in \mathcal{B}$  and  $\xi \in \cup_{m \leq 200} \Xi_m$ , where  $\Xi_m$  was defined in (19). The cardinality of this set is given by

$$\# \bigcup_{m \leq 200} \Xi_m = \sum_{m \leq 200} \left( \sum_{d|m} d^3 \mu(m/d) \right) = 373111960$$

where  $\mu$  is the Möbius function. (The divisor sum gives the number of lattice points in  $\Xi_m$  that do not belong to any lattice of smallest index.) Now, every fractional part  $\xi$  corresponds to at least one cycle, so that the difference between the number of periodic points and the number of their fractional parts gives the size of the boundary set, restricted to these denominators (to match the figures, one has to take into account the fact that two  $\xi$ -values correspond to three and four  $\beta$ -values, respectively). In the present data set, the boundary points are one in  $10^6$ , a vanishing fraction of the total. This figure is smaller than the corresponding data for orbits with fixed period (comparison requires adjusting for the different size of the two data samples, see remarks following table 2). These findings underpin our conjecture 1.

## References

- [1] S. Akiyama, Self-affine tiling and Pisot numeration system. In *Number theory and its applications*, S. Kanemitsu and K. Györy (eds.) Kluwer Academic Publishers, The

- Netherlands (1999) 7–17.
- [2] S. Akiyama, T. Borbéli, H. Brunotte, A. Pethö, J. Thuswaldner, Generalized radix representations and dynamical systems I, *Acta Math. Hungarica* **108** (2005) 207–238.
  - [3] S. Akiyama, H. Brunotte, A. Pethö, J. Thuswaldner, Generalized radix representations and dynamical systems II, *Acta Arith.* **121** (2006), 21–61.
  - [4] Arnoux P. and Yoccoz, J., 1981, Construction de difféomorphismes pseudo-Anosov, *C. R. Acad. Sci. Paris* **292** 75–78.
  - [5] M. Bartuccelli and F. Vivaldi, Ideal orbits of toral automorphisms, *Physica D* **39** (1989) 194–204.
  - [6] A. Bertrand, Développement en base de Pisot et répartition modulo 1, *C. R. Acad. Sc. Paris* **385** (1977) 419–421.
  - [7] D. Bosio and F. Vivaldi, Round-off errors and  $p$ -adic numbers, *Nonlinearity* **13** (2000) 309–322.
  - [8] J. Buzzi, Piecewise isometries have zero topological entropy, *Ergod. Th. and Dynam. Sys.* **21** (2001) 1371–1377.
  - [9] B.V. Chirikov and F. Vivaldi, An algorithmic view of pseudochaos, *Physica D* **129** (1999) 223–235.
  - [10] H. Cohen, A course in computational algebraic number theory, Springer-Verlag, Berlin (1996).
  - [11] M. Degli Esposti and S. Isola, Distribution of closed orbits for linear automorphisms of tori, *Nonlinearity* **8** (1995) 827–842.
  - [12] W. J. Gilbert, Fractal geometry derived from complex bases, *Math. Intell.* **4** (1982) 78–86.
  - [13] R. Hartley and T. O. Hawkes, *Rings, modules and linear algebra*, Chapman and Hall, London (1970).
  - [14] A. Katok and B Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge (1997).
  - [15] R. Kenyon and A. Vershik, Arithmetic construction of sofic partitions and hyperbolic toral automorphisms, *Ergod. Theory and Dyn. Sys.* **18** (1998) 357–372.
  - [16] K. L. Kouptsov, J. H. Lowenstein, and F. Vivaldi, Quadratic rational rotations of the torus and dual lattice maps, *Nonlinearity* **15** (2002) 1795–1482.
  - [17] J. H. Lowenstein and F. Vivaldi, Embedding dynamics for round-off errors near a periodic orbit, *Chaos* **10** (2000) 747–755.
  - [18] J. H. Lowenstein, K. L. Kouptsov and F. Vivaldi, Recursive tiling and geometry of piecewise rotations by  $\pi/7$ , *Nonlinearity* **17** (2004) 1–25.
  - [19] J. H. Lowenstein, G. Poggiaspalla and F. Vivaldi, Sticky orbits in a kicked-oscillator model, *Dynamical Systems* **20** (2005) 413–451.

- [20] J. H. Lowenstein and G. Poggiaspalla and F. Vivaldi, Interval exchange transformations over algebraic number fields: the cubic Arnoux-Yoccoz model, *Dynamical Systems* **22** (2007) 73–106. <http://maths.qmul.ac.uk/~fv/research/AY.pdf>.
- [21] D. A. Marcus, *Number fields*, Springer-Verlag, New York (1977).
- [22] H. Möller, Zur Verteilung der Restindizes Ganzer Zahlen, Pershl Festband, Hamburg (1972).
- [23] I. Niven, *Irrational numbers*, The Mathematical Association of America, Washington DC (1956).
- [24] I. C. Percival and F. Vivaldi, Arithmetical properties of strongly chaotic motions, *Physica D* **25** (1987) 105–130.
- [25] G Poggiaspalla, J H Lowenstein and F Vivaldi, Geometric representation of interval exchange maps over algebraic number fields *Nonlinearity* **21** (2008) 149–177.
- [26] M. Pohst and H. Zassenhaus, *Algorithmic algebraic number theory*, Cambridge University Press, Cambridge (1990).
- [27] M. RamMurty, Artin’s conjecture for primitive roots, *Math. Intelligencer* **10** (1988) 59–67.
- [28] K. Schmidt, On periodic expansions of Pisot numbers and Salem number, *Bull. London Math. Soc.* **12** (1980) 269–278.
- [29] N. Sidorov, Arithmetic dynamics, in *Topics in dynamics and ergodic theory*, LMS Lecture Notes Series **310** (2003) 145–189.
- [30] F. Vivaldi, The arithmetic of discretized rotations, in *p-adic Mathematical Physics*, A. Y. Khrennikov, Z. Rakic, I. V. Volovich editors, AIP Conference Proceedings No. 826 AIP, Melville, New York (2006) 162–173.