Arithmetical properties of a family of irrational piecewise rotations

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Abstract

We study a family of piecewise rotations of the torus with irrational rotation number, depending on a parameter λ. Our approach is arithmetical. We represent periodic coordinates explicitly as elements of the rational function field $\mathbb{Q}(\lambda)$. A similar representation is derived for the points that recur to the boundary of the atoms, which we call *pseudo-hyperbolic* points. Using a uniqueness property of these points, established via non-archimedean methods, we prove that for transcendental or rational values of the parameter λ, our map has no unstable periodic orbits. By contrast, unstable cycles do exist for parameter values which are algebraic numbers of degree greater than one, and they represent bifurcations. For rational values of λ with prime-power denominator, we show that, in the p-adic metric, all rational bounded orbits are periodic. We investigate experimentally some asymptotic properties of periodic orbits in relation to a conjecture by Ashwin that the Lebesgue measure of the closure of the discontinuity set is positive. In this context, we find an unexpected absence of non-symmetric periodic orbits.

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1 Introduction

Let S be the matrix

$$S = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \qquad \lambda = 2\cos(2\pi\theta) \tag{1}$$

and let $\Omega = [0,1)^2$ be the unit square —a fundamental domain of the lattice $\mathbb{Z}^2$. The *torus map* generated by the triple $(S, \mathbb{Z}^2, \Omega)$ is defined as [25]

$$L : \Omega \to \Omega \qquad z \mapsto (Sz - \mathbb{Z}^2) \cap \Omega. \tag{2}$$

Because $S\Omega$ also tiles the plane under translation, the map L is invertible[1], and features a surprisingly rich dynamics, from minimal ingredients; it is non-ergodic, and has zero topological entropy [9]. (By contrast, the parameter values $\lambda \in \mathbb{Z}$, $|\lambda| > 2$ lead to algebraic automorphisms of the 2-torus, which are ergodic and have positive entropy [27].) The example shown in figure 1 refers to the rational parameter $\lambda = 1/2$, corresponding to an *irrational* value of the rotation number θ.

The map L is linearly conjugate to a piecewise rotation on a rhombus with rotation number θ. This is an example of a *piecewise isometry,* a class of dynamical systems that generalize to higher dimensions the notion of interval exchange maps; such systems are now object of intense investigation [2, 4, 8, 14–16, 19, 23, 25]. Much literature is devoted to the case of rational rotation number, for which the stable regions in phase space — the ellipses of figure 1— are convex polygons. In the representation (1), the parameters λ corresponding to a rational rotation number are a special type of algebraic numbers —twice the real part of roots of unity. The case of quadratic λ has been thoroughly studied, exploiting the presence of exact scaling of orbits (see aforementioned literature, and also some applications to the dynamics of round-off errors [28, 31]). The cubic case proved more difficult; there are some exact results on specific models, and substantial experimentation [19, 24, 29, 30, 33]. Rational rotation numbers with prime denominator were considered in [18] from a ring-theoretic angle, in a rather general setting. In all cases in which computations have been performed, the complement of the cells, namely the closure of the so-called *discontinuity set*, has been found to have zero Lebesgue measure.

The case of *irrational rotation number* —the generic one— has attracted comparatively less interest. Variants of the map L with irrational rotation number were studied extensively since the late 1980's, as models of second-order nonlinear digital filters [5, 10, 11, 13, 39]. While studying a specific family of irrational piecewise isometries, Ashwin conjectured that the closure of the discontinuity set has positive measure [3], and that this measure depends continuously on the parameter. Subsequently, the semi-continuous parameter dependence was established rigorously [17]; however, the important question of positivity of measure remains unresolved.

The goal of this paper is to study the map (2) for irrational rotation numbers θ, using an arithmetical approach. The generic case corresponds to *transcendental* values of λ, which have full Lebesgue measure. Arithmetically, this amounts to regarding λ as an indeterminate, leading to the rational function field over $\mathbb{Q}$. Any *rational* parameter value

[1] S is a *parquet matrix* for $\mathbb{Z}^2$ and Ω —see [2]

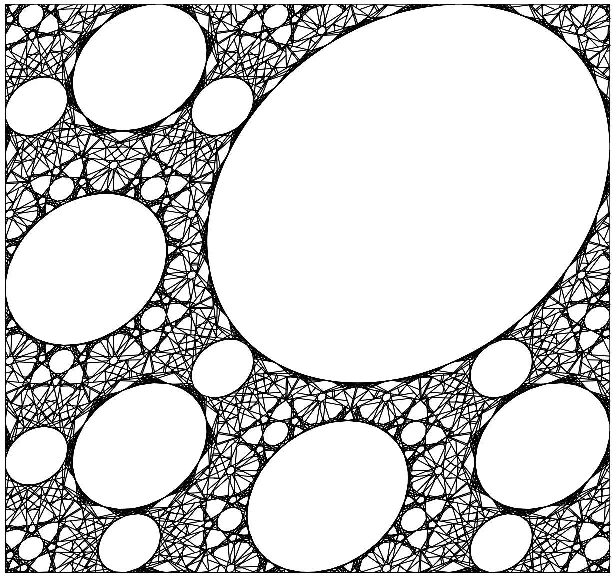

Figure 1: Partial construction of the discontinuity set, for $\lambda = 1/2$, involving 80 forward and backward images of the generator Γ_0. We recognize elliptical cells, supporting quasiperiodic motion, and a periodic orbit at the centre. There are regions surrounding cells where the images of Γ_0 accumulate rapidly, and regions where they accumulate slowly, possibly reflecting the presence of invariant curves.

which is not an integer also corresponds to an irrational rotation: we shall exploit non-archimedean techniques to deal with the transcendental and rational cases with a unified formalism. The case of non-rational *algebraic* values of λ is more complex, and it will be the subject of a future investigation. In this paper we merely identify a relevant family of finite algebraic extensions associated to bifurcations of periodic orbits.

The material is structured as follows. In section 2, after some basic definitions, we develop the standard constructs related to the map's time-reversal symmetry, which is to be exploited throughout the paper.

Explicit formulae for the coordinates of periodic points, as elements of the rational function field $\mathbb{Q}(\lambda)$, are established in section 3 (theorem 2); these rational functions are specified by the symbolic dynamics, and are expressed in terms of a distinguished family of irreducible polynomials. These functions may be specialized to any parameter value corresponding to a rotation number θ which is either irrational, or rational with denominator not dividing the period; they are also generalizable to any symmetric domain Ω. It turns out that unstable periodic orbits —that is, orbits on the discontinuity set[2]— do not exist for generic parameter values (see below). By contrast, such orbits can exist when the parameter λ is an algebraic number, and we indicate how they are constructed as bifurcation parameters corresponding to the disappearance of stable cycles. Such bifurcation points are boundary points of parametric intervals for which a cycle of a given code exists. The period of the unstable orbits is constrained arithmetically by the following result, valid for

[2]If the periodic points are not isolated, this definition of instability is weaker than that given in [35, p. 183], as it would correspond to the so-called mixed type.

any rotation number (theorem 4)

Theorem A. *If the parameter λ is an algebraic number of degree d, then the map L has no unstable periodic orbits of period less than $2d$.*

In section 4 we develop the notion of *pseudo-hyperbolic points,* which, while typically non-periodic —and indeed non-hyperbolic— nonetheless bear many useful analogies to unstable periodic points. The pseudo-hyperbolic points are the recurrent points of the boundary of the atoms; geometrically, they correspond to transversal intersections of two segments of the discontinuity set, which play the role of separatrices in smooth systems. The pseudo-hyperbolic points are arranged into finite sequences: as for periodic orbits, these sequences are represented by rational functions of the parameter λ, determined by a *finite* symbolic code (theorem 6).

Studying periodic and pseudo-hyperbolic orbits from a valuation-theoretic angle unveils further similarities between them (section 5). The following result (theorem 8), proved using non-archimedean methods, illustrates a typical interplay between dynamics and arithmetic

Theorem B. *If the parameter λ is transcendental or rational, then any orbit of the map L contains at most one pseudo-hyperbolic sequence; in particular, there are no unstable periodic orbits. (The fixed point at the origin is excluded from consideration.)*

Since transcendental numbers have full Lebesgue measure, the absence of unstable cycles is generic. The situation contrasts markedly with that of rational rotation numbers, where the discontinuity set exhibits a rich periodic structure [25, proposition 6.2]. By contrast, it is not clear whether or not maps with irrational rotation number exists which have infinitely many unstable periodic orbits.

The next result (extracted from theorem 7 and lemma 10 of section 5) characterizes periodic motions in the non-archimedean metric

Theorem C. *Let λ be a rational number with prime-power denominator p^n. Then a rational orbit of L is periodic if and only if it is contained in a p-adic disk of radius p^{-n}.*

The rational orbits of the map (2) in the p-adic metric behave as if they were in presence of unstable equilibria. It turns out that the euclidean rotation given by the matrix S becomes hyperbolic on the p-adic plane, and consequently all periodic orbits are hyperbolic. The emergence of hyperbolicity/expansiveness in systems with discrete phase space, equipped with a non-archimedean metric, is not new. For example, with reference to equation (2), we note that the *lattice map* generated by the same triple $(S, \mathbb{Z}^2, \Omega)$, namely

$$\hat{L} : \mathbb{Z}^2 \to \mathbb{Z}^2 \qquad z \mapsto (Sz - \Omega) \cap \mathbb{Z}^2 \tag{3}$$

can be embedded into a smooth expanding map of the p-adic integers [7]. Superficially, the maps L and $\hat{L}$ exhibit a form of duality; in fact, they are quite unrelated, and in particular no smooth deterministic embedding system exists for $\hat{L}$: in section 5 we will argue that a natural p-adic embedding of L must have a stochastic element.

Finally, in section 6, we describe the results of accurate numerical experiments on periodic orbits asymptotics. At two specific rational parameter values, we compute all cycles with period up to 10^7, together with the area of the corresponding cells. The measure of the periodic set appears to approach a proper fraction of the total measure, with an exponential convergence rate. One experiment involved a variant of the map L considered by Ashwin [3]: our results agree with his (which were obtained by a box-counting estimate of the complementary set), and improve the accuracy of an associated exponent.

Of note is the fact that we did not find any non-symmetric cycles at all, while in a related model (discussed in section 6.1) we were able to justify only the existence of *finitely many* of them. These experimental findings contrast with the case of reversible maps with a sufficiently rich symbolic dynamics, where one would expect non-symmetric periodic orbits to dominate the statistics for large periods. (We are not aware of explicit results on this phenomenon, which, however, appears to be 'well-known'. Symmetry is reflected in the symbolic dynamics (cf. proposition 3, section 3), and if there are enough symbol sequences —e.g., in the case of a full shift— most of them will not be symmetric[3]. Alternatively, one may examine the factorization of dynamical zeta functions [12].) The scarcity of non-symmetric orbits was noted recently in the quite different context of maps over finite fields [34], where the obstruction to their existence was found to be probabilistic in nature. In the present case, identifying the root cause of this phenomenon could shed light on the problem of the measure of the exceptional set.

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2 Discontinuity and reversibility

Letting $z = (x, y) \in \Omega$, one verifies that the map L is given explicitly by

$$L: \; (x,y) \mapsto (\lambda x - y + \iota(x,y), x) \qquad \iota(x,y) = -\lfloor \lambda x - y \rfloor, \tag{4}$$

where $\lfloor\cdot\rfloor$ denotes the floor function. The quantity ι takes a finite set of integers values, which depend on λ as follows

λ	$\iota(\Omega)$
$1 < \lambda < 2$	$\{-1, 0, 1\}$
$0 < \lambda < 1$	$\{0, 1\}$
$-1 < \lambda < 0$	$\{1, 2\}$
$-2 < \lambda < -1$	$\{1, 2, 3\}$

(5)

[3]We are grateful to J S W Lamb for this observation.

The level sets of the function ι are the *atoms* Ω_i, which are convex polygons; from the table we see that there are two atoms for $|\lambda| < 1$ and three atoms for $|\lambda| > 1$. For $\lambda < 0$, the origin $(0, 0)$ constitutes an additional atom, corresponding to the value $\iota = 0$: this point will be excluded from consideration. Given a code[4] $\iota = (\iota_0, \iota_1, \ldots)$, with integer symbols ι_t taken from the alphabet (5), we consider the set $\mathcal{C}(\iota)$ of the points $z \in \Omega$ for which $\iota(L^t(z)) = \iota_t$, for $t = 0, 1, \ldots$. These are the points whose images visit the atoms in the order specified by the code. If $\mathcal{C}(\iota)$ is non-empty, then it is called a *cell.* A cell having positive measure consists of an open ellipse together with a subset of its boundary [17, proposition 2]. In coordinates relative to their centre, these ellipses are similar to the ellipse $x^2 - \lambda xy + y^2 = 1$.

The set of all images and pre-images of the boundary of the atoms constitutes the *discontinuity set* Γ

$$\Gamma = \bigcup_{t=-\infty}^{\infty} L^t(\partial\Omega) \qquad\qquad \partial\Omega = \bigcup_i \partial\Omega_i \tag{6}$$

which consists of a countable set of segments. The set $\partial\Omega$ is not necessarily minimal, in the sense that Γ may also be generated by iterating a proper subset of $\partial\Omega$. For our purpose it is important to construct generating sets with certain minimality properties. Thus we say that a subset Γ_0 of $\partial\Omega$ is a *generator* of the discontinuity set if:

(i) $\Gamma = \bigcup_t L^t(\Gamma_0)$;

(ii) $\Gamma_0 = \{\gamma\}$ is a set of segments γ with the property that for all $\gamma, \gamma' \in \Gamma_0$, and $t, t' \in \mathbb{Z}$, neither the union, nor the intersection of $L^t(\gamma)$ and $L^{t'}(\gamma')$ is a segment.

In other words, the number of segments of Γ_0 is minimal, and no segment can be deleted from Γ_0. As a result, the set of recurrent points

$$\Gamma_0 \bigcap \left(\bigcup_{t=1}^{\infty} L^{\mp t}(\Gamma_0) \right) \tag{7}$$

is countable. The existence of a nontrivial intersection (7), connected to the presence of pseudo-hyperbolic points (see section 4), makes the construction of a minimal generating set unworkable. By contrast, for an irrational rotation number, the condition *(ii)* is effectively computable.

The segment Γ_0, with endpoints $(0, 0)$ (included) and $(1, 0)$ (excluded) is a generator of Γ. To see this, we notice that Γ_0, together with its pre-image (the segment with endpoints $(0, 0)$ and $(0, 1)$), comprise the boundary of the square Ω, from which the boundary of the atoms are obtained with a single iteration. Thus Γ_0 generates Γ. Locally, L is conjugate to an irrational rotation followed by a translation, so that all images of Γ_0 have a distinct orientation, and hence *(ii)* is satisfied.

In figure 1 we display a partial construction of Γ obtained iterating the generator Γ_0 forward and backward 80 times. If $\mathcal{O}$ is a periodic orbit, then either $\mathcal{O} \subset \Gamma$ or $\mathcal{O}$ lies

[4] For economy of notation, we use the symbol ι for both coding function and code.

at finite Hausdorff distance from Γ. In the latter case it is easy to show that the cell surrounding each point z of $\mathcal{O}$ has area

$$\mathcal{A} = \frac{d^2\pi}{2}\sqrt{4-\lambda^2} \qquad d = \min_{z\in\mathcal{O}}[z] \tag{8}$$

where $[z]$ is the minimum distance of the coordinates of z from an integer. Thus, if a periodic point is computable exactly (for instance, for algebraic values of the parameter λ), so is the area of its cell.

The equation

$$L^{-1} = G \circ L \circ G^{-1} \qquad G : (x, y) \mapsto (y, x) \tag{9}$$

shows that map L is reversible [26], and hence can be written as a product of two orientation-reversing involutions[5]

$$L = H \circ G \qquad H : (x, y) \mapsto (\{-x + \lambda y\}, y) \tag{10}$$

where $\{\cdot\}$ denotes the fractional part. Repeated use of (9) yields the useful formula

$$L^{-t} = G \circ L^t \circ G \qquad t \in \mathbb{Z}. \tag{11}$$

Let Fix(G) and Fix(H) be the set of points left invariant by G and H, respectively. The set Fix(G) is the half-open segment with endpoints $(0, 0)$ (included) and $(1, 1)$ (excluded). It can be verified that Fix(H) is the union of segments $z_k z'_k$ where

λ	z_1	z'_1	z_2	z'_2	z_3	z'_3
$1 < \lambda < 2$	$(0, 0)$	$(\frac{\lambda}{2}, 1)$	$(\frac{1}{2}, 0)$	$(1, \frac{1}{\lambda})$	$(0, \frac{1}{\lambda})$	$(\frac{\lambda-1}{2}, 1)$
$0 < \lambda < 1$	$(0, 0)$	$(\frac{\lambda}{2}, 1)$	$(\frac{1}{2}, 0)$	$(\frac{\lambda+1}{2}, 1)$		
$-1 < \lambda < 0$	$(\frac{1}{2}, 0)$	$(\frac{\lambda+1}{2}, 1)$	$(1, 0)$	$(\frac{\lambda+2}{2}, 1)$		
$-2 < \lambda < -1$	$(\frac{1}{2}, 0)$	$(0, -\frac{1}{\lambda})$	$(1, 0)$	$(\frac{\lambda+2}{2}, 1)$	$(1, -\frac{1}{\lambda})$	$(\frac{\lambda+3}{2}, 1)$

Table 1: The set Fix(H), consisting of the half-open segments $z_k z'_k$; the endpoint z'_k is excluded.

Fix(G) and Fix(H) are called the *symmetry lines* of L; they are not defined uniquely, since they could be replaced by their image under any iterate of the map L. Letting $z_t = (x_t, x_{t-1}) \in \Omega$ and $L(z_t) = z_{t+1}$, we derive the useful characterizations

$$z_t \in \text{Fix}(G) \iff x_t = x_{t-1} \qquad z_t \in \text{Fix}(H) \iff x_t = x_{t-2}. \tag{12}$$

An orbit $\mathcal{O}$ is *symmetric* if $G(\mathcal{O}) = \mathcal{O}$; this condition holds if it holds at a single point in the orbit. It can be shown that a symmetric orbit has one point on Fix(G) or on Fix(H). Furthermore, a symmetric *periodic* orbit of odd period $2n+1$ will have exactly one point z in Fix(G) and one point $L^n(z)$ in Fix(H); a symmetric orbit of even period $2n$ will have two points on Fix(G) (z and $L^n(z)$) and none on Fix(H), or vice-versa [26, section 4.1]. From the above it follows that the symmetric periodic orbits can be computed from

[5]meaning that $G^2 = H^2 = \text{Id}$, and $\det(dG) = \det(dH) = -1$.

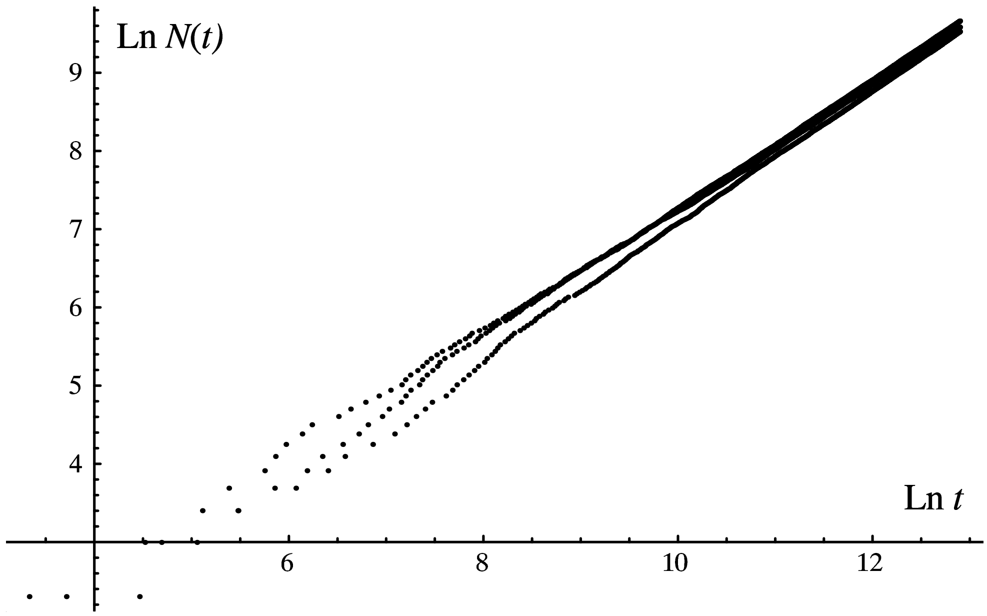

Figure 2: Log-log plot of $N(t)$, the number of symmetric periodic orbits of period not exceeding t, for $t \leq 400000$. For each increment of 10 in $N(t)$, we plot a data point for each of the λ values $-1/2, -3/2, 1/3$, in left-to-right order. For $40000 < t < 400000$, $N(t)$ is approximately proportional to $t^{\delta(\lambda)}$, with $\delta(-3/2) = 0.83$, $\delta(-1/2) = 0.85$, $\delta(1/3) = 0.82$.

the intersections of the images of $\mathrm{Fix}(G)$ and $\mathrm{Fix}(H)$. Computing non-symmetric periodic orbits is considerably more difficult: numerical experiments show that such orbits do exist, but are very rare (see section 6).

An example of growth of the number of symmetric periodic orbits with the period is displayed in figure 2. Besides the compelling inference that the number of cycles is infinite, the approximately linear growth rate suggests that the set of allowed periods has positive density in $\mathbb{N}$.

3 Periodic points

In this section we derive explicit formulae for periodic points, as rational functions of the parameter λ, which are specified by the code ι (theorem 2). The parameter λ plays the role of an indeterminate, and it can be specialized to any value corresponding to an irrational rotation number, and also to rational rotation numbers whose denominator does not divide the period.

Before dealing with periodic orbits, we deal with iterates

Lemma 1 *Let $z = (x_0, x_{-1})$ have code $\iota = (\iota_0, \iota_1, \ldots)$, and let $L^t(z) = (x_t, x_{t-1})$. Then* *for* $t = 0, 1, \ldots,$ *we have*

$$x_t = x_0\, S_t(\lambda) - x_{-1}\, S_{t-1}(\lambda) + R_{t-1}(\lambda)$$

where $S_t(\lambda) \in \mathbb{Z}[\lambda]$ *is a monic polynomial of degree* t, *and*

$$S_{-1} = 0 \quad S_0 = 1 \qquad\qquad S_{t+1}(\lambda) = \lambda S_t(\lambda) - S_{t-1}(\lambda) \tag{13}$$

$$R_{-1} = 0 \qquad\qquad R_t(\lambda) = \sum_{k=0}^{t} \iota_k\, S_{t-k}(\lambda). \tag{14}$$

The polynomials $S_t(\lambda)$ are Chebyshev polynomials of the second kind [1, chapter 22]. The proof of the formulae is a straightforward induction, and will be omitted. The degree of $R_t(\lambda)$ depends on the code, is at most t, and orbits exists for which it is exactly t, corresponding to $\iota_0 \neq 0$. We shall use repeatedly the following identity, proved in the appendix

$$S_{t-1}S_{t-k} - S_t S_{t-1-k} = S_{k-1} \qquad k, z \in \mathbb{Z}. \tag{15}$$

The use of negative indices in this formula is justified by the fact that the recursion relation (13) can be inverted, to give

$$S_{-t} = -S_{t-2} \qquad t \in \mathbb{Z}. \tag{16}$$

Equation 4 shows that the set

$$\Omega_\lambda = \mathbb{Q}(\lambda)^2 \cap \Omega \tag{17}$$

of the points in Ω with coordinates in $\mathbb{Q}(\lambda)$ is invariant under the map L. This set contains the endpoints of the segments comprising the discontinuity set Γ; this is because the generator Γ_0 has endpoints in Ω_λ, and the effect of the discontinuity can only produce points within Ω_λ, as easily verified. It turns out that Ω_λ also contains the periodic points (theorem 2), and the pseudo-hyperbolic points (theorem 6).

To derive periodic points formulae, we shall need the polynomials

$$\Psi_1(x) = x - 2, \qquad \Psi_2(x) = x + 2, \qquad \Psi_t(y + y^{-1}) = F_t(y) y^{-\phi(t)/2} \qquad t > 2, \tag{18}$$

where $F_t(x)$ is the t-th cyclotomic polynomial[6] and ϕ is Euler's function [32, p 37]. For $t > 2$, Ψ_t is a monic polynomial in $x = y + y^{-1}$, of degree $\phi(t)/2$. Moreover, Ψ_t is irreducible for all t, and its roots are the distinct numbers $2\cos(2\pi k/t)$, with k coprime to t. These properties of Ψ_t are established from the fact that the polynomial F_t has degree $\phi(t)$, is irreducible and reflexive[7], together with the repeated use of the identity

$$x^k + x^{-k} = (x + x^{-1})(x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k}) \qquad k > 0.$$

We now define define four sequences of polynomials, which admit factorization in terms of the Ψ-polynomials.

$$\mathcal{M}_t(x) = \prod_{d|t} \Psi_d(x) = \prod_{k=0}^{\lfloor t/2 \rfloor} (x - 2\cos(2\pi k/t)) \qquad t = 1, 2, \ldots \tag{19}$$

[6]the roots of F_t are the primitive t-th roots of unity

[7]meaning that $x^{\phi(t)} F_t(x^{-1}) = F_t(x)$

and

$$\mathcal{M}'_1 = \mathcal{M}'_2 = 1 \qquad \mathcal{M}'_t(x) = \prod_{\substack{d|t\\ d>2}} \Psi_d(x) \qquad t = 3, 4, \ldots . \tag{20}$$

Using the rightmost expression in (19), or, equivalently, the formula $\sum_{d|t} \phi(d) = t$ (see [21, theorem 63]), we find that the degree of $\mathcal{M}_t$ is equal to $(t+1)/2$ if t is odd, and to $(t+2)/2$ if t is even, while the degree of $\mathcal{M}'_t$ is equal to $(t-1)/2$ and $(t-2)/2$, respectively.

Next we define

$$\mathcal{W}_0 = 1 \qquad \mathcal{W}_k(x) = \prod_{\substack{d|2k+1\\ d\neq 1}} \Psi_{2d}(x), \quad k = 1, 2, \ldots \tag{21}$$

$$\mathcal{V}_0 = 1 \qquad \mathcal{V}_k(x) = \prod_{d|\frac{k}{e}} \Psi_{4ed}(x), \quad k = 1, 2, \ldots , \tag{22}$$

where e is the largest power of 2 dividing k. By construction, $\mathcal{W}_k$ and $\mathcal{V}_k$ are monic polynomials with integer coefficients. Their degree is equal to k; to prove this for $\mathcal{W}_k$ we note that

$$\sum_{\substack{d|2k+1\\ d\neq 1}} \phi(2d) = \sum_{\substack{d|2k+1\\ d\neq 1}} \phi(2)\phi(d) = \sum_{\substack{d|2k+1\\ d\neq 1}} \phi(d) = 2k, \tag{23}$$

whereas for $\mathcal{V}_k$ we have the identity

$$\sum_{d|\frac{k}{e}} \phi(4ed) = \sum_{d|\frac{k}{e}} \phi(4e)\phi(d) = \sum_{d|\frac{k}{e}} 2e\phi(d) = 2e\frac{k}{e} = 2k. \tag{24}$$

The first connection between the map L and the polynomials introduced above is provided by the following identity in $\mathbb{Z}[\lambda]$, which we prove in the appendix

$$S_t(\lambda) = \mathcal{M}'_{2t+2}(\lambda) \qquad t = 0, 1, \ldots . \tag{25}$$

We now state and prove the main result of this section, the periodic point formulae. The input datum is an integer periodic code ι, the output is a pair or rational functions in λ, expressed in terms of the Ψ-polynomials (18), which describe the periodic point with code ι, as the parameter is varied. Not all codes give actual orbits though —see below.

Theorem 2 *Let $z = (x, y)$ be a periodic point of L, with period t and code $\iota = (\overline{\iota_0, \iota_1, \ldots, \iota_{t-1}})$. If the rotation number is irrational, or rational with denominator not divisible by t, then*

$$x = -\frac{\mathcal{X}_t(\lambda, \iota)}{\mathcal{M}_t(\lambda)} \qquad y = -\frac{\mathcal{X}_t(\lambda, \sigma^{-1}(\iota))}{\mathcal{M}_t(\lambda)} \tag{26}$$

where $\mathcal{M}_t$ is given by (19), σ is the shift map, and the polynomial $\mathcal{X}_t \in \mathbb{Z}[\lambda]$ is given by

$$\mathcal{X}_{2n+1}(\lambda, \iota) = \iota_0 \mathcal{W}_n(\lambda) + \sum_{k=1}^{n} (\iota_k + \iota_{2n+1-k})\, \mathcal{W}_{n-k}(\lambda) \qquad n = 0, 1, \ldots \tag{27}$$

$$\mathcal{X}_{2n}(\lambda, \iota) = \iota_0 \mathcal{V}_n(\lambda) + \sum_{k=1}^{n} (\iota_k + \iota_{2n-k})\, \mathcal{V}_{n-k}(\lambda) \qquad n = 1, 2, \ldots , \tag{28}$$

with $\mathcal{W}_k$ and $\mathcal{V}_k$ as defined in (21) and (22), respectively.

Proof. If (x, y) is a t-cycle, then, letting $x = x_t = x_0$ and $y = x_{t-1} = x_{-1}$ in lemma 1, we find

$$x = \frac{X_t}{D_t} \qquad \text{with} \qquad X_t = (1 + S_{t-2})R_{t-1} - S_{t-1}R_{t-2} \tag{29}$$

and

$$D_t = (1 - S_t)(1 + S_{t-2}) + S_{t-1}^2 = -S_t + S_{t-2} + 2. \tag{30}$$

The rightmost equality follows from equation (15) with $k = 1$. Now, $S_t - S_{t-2}$ is the trace of the matrix S^t given in equation (59) of the appendix, and because the latter for $|\lambda| < 2$ is conjugate to a rotation by $2\pi t\theta$, we have that $D_t = 2 - 2\cos(2\pi t\theta)$. In particular, $D_t \neq 0$ for all irrational θ, and also for the rational θ whose denominator does not divide t.

The equation $D_t(\lambda) = 0$ gives $\cos(2\pi t\theta) = 1$, hence $\theta = k/t$, $k \in \mathbb{Z}$, that is, $\lambda = \lambda_k = 2\cos(2\pi k/t)$, $t = 0, \ldots, \lfloor t/2 \rfloor$, We find

$$\left.\frac{dD_t(\lambda)}{d\lambda}\right|_{\lambda_k} = -2t\,\frac{\sin(2\pi k)}{\sqrt{4 - \lambda_k^2}} = 0 \qquad\qquad k \neq 0,\, t/2$$

and one verifies that for $k = 0, t/2$ the above derivative is non-zero. Taking into account the fact that $D_t(\lambda)$ has degree t, we conclude that its roots are double, apart from $\lambda = \pm 2$ which are simple.

Now, from the above, and equations (19) and (20) it follows that the polynomial $-\mathcal{M}_t\mathcal{M}'_t \in \mathbb{Z}[\lambda]$ has the same degree as $D_t(\lambda)$, the same leading coefficient, and the same roots with the same multiplicity. Thus $D_t = -\mathcal{M}_t\mathcal{M}'_t$.

Using lemma 1 and the identity (15), we transform the quantity X_t defined in equation (29), as follows

$$\begin{aligned}
X_t &= R_{t-1} + S_{t-2}R_{t-1} - S_{t-1}R_{t-2} \\
&= R_{t-1} + \sum_{k=0}^{t-1} \iota_k S_{t-2}S_{t-1-k} - \sum_{k=0}^{t-2} \iota_k S_{t-1}S_{t-2-k} \\
&= R_{t-1} + \iota_{t-1}S_{t-2} + \sum_{k=0}^{t-2} \iota_k \left(S_{t-2}S_{t-1-k} - S_{t-1}S_{t-2-k}\right) \\
&= R_{t-1} + \iota_{t-1}S_{t-2} + \sum_{k=0}^{t-2} \iota_k\, S_{k-1} \\
&= \sum_{k=0}^{t-1} \iota_k (S_{t-1-k} + S_{k-1}).
\end{aligned} \tag{31}$$

Defining

$$P_{t,k} = S_{t-1-k} + S_{k-1}, \tag{32}$$

and recalling the definitions (21) and (22) we shall prove the following factorization properties, valid in $\mathbb{Z}[\lambda]$

$$P_{2n+1,k} = \mathcal{M}'_{2n+1}\mathcal{W}_{n-k} \qquad n = 0, 1, \ldots \tag{33}$$
$$P_{2n,k} = \mathcal{M}'_{2n}\mathcal{V}_{n-k} \qquad n = 1, 2, \ldots. \tag{34}$$

Using the definition of Ψ given in equation (18), equation (25), and the divisor product

$$\prod_{d|t} F_d(y) = y^t - 1 \tag{35}$$

with $y = \lambda + \lambda^{-1}$, we transform (32) as follows

$$\begin{aligned} P_{t,k}(\lambda) &= \mathcal{M}'_{2t-2k}(\lambda) + \mathcal{M}'_{2k}(\lambda) \\ &= \prod_{\substack{d|2t-2k \\ d>2}} \Psi_d(\lambda) + \prod_{\substack{d|2k \\ d>2}} \Psi_d(\lambda) \\ &= \prod_{\substack{d|2t-2k \\ d>2}} F_d(y) y^{-\phi(d)/2} + \prod_{\substack{d|2k \\ d>2}} F_d(y) y^{-\phi(d)/2} \\ &= \frac{1}{y^2-1}\left[y^{-(2t-2k-2)/2}(y^{2t-2k}-1) + y^{-(2k-2)/2}(y^{2k}-1)\right] \\ &= \frac{1}{y^2-1}\left(y^{t-k+1} - y^{-t+k+1} + y^{k+1} - y^{-k+1}\right). \end{aligned} \tag{36}$$

We now prove (33). Let $t = 2n+1$; we shall make use of the following identity

$$\begin{aligned} \frac{y^{2n+1}+1}{y+1} &= \frac{y^{2(2n+1)}-1}{(y+1)(y^{2n+1}-1)} = \frac{\prod_{d|2(2n+1)} F_d(y)}{F_2(y)\prod_{d|2n+1} F_d(y)} \\ &= \prod_{\substack{d|2(2n+1) \\ d\not| 2n+1 \\ d\neq 2}} F_d(y) = \prod_{\substack{d|2n+1 \\ d\neq 1}} F_{2d}(y). \end{aligned}$$

The last equality follows from the fact that the required divisors must be even (lest they divide $2n+1$), and that every even divisor of $2(2n+1)$ contributes to the product. Hence such divisors are twice the divisors of $2n+1$. Using this, together with (23), we transform the RHS of (33) as follows

$$\begin{aligned} \mathcal{M}'_{2n+1}(\lambda)\mathcal{W}_{n-k}(\lambda) &= \prod_{\substack{d|2n+1 \\ d\neq 1}} \Psi_d(\lambda) \prod_{\substack{d|2(n-k)+1 \\ d\neq 1}} \Psi_{2d}(\lambda) \\ &= \prod_{\substack{d|2n+1 \\ d\neq 1}} F_d(y) y^{-\phi(d)/2} \prod_{\substack{d|2(n-k)+1 \\ d\neq 1}} F_{2d}(y) y^{-\phi(2d)/2} \\ &= \frac{1}{y^2-1} y^{-(2n+1-1)/2}\left(y^{2n+1}-1\right) y^{-(2n-2k+1-1)/2}\left(y^{2n-2k+1}+1\right) \\ &= \frac{1}{y^2-1}\left(y^{2n-k+2} + y^{k+1} - y^{-k+1} - y^{-2n+k}\right) \end{aligned}$$

and one verifies that the last expression coincides with (36) with $t = 2n+1$, as desired.

Next we prove (34). Let n be a positive integer; if e is the largest power of 2 dividing n, we have

$$y^{2n}+1=\frac{y^{4n}-1}{y^{2n}-1}=\frac{\prod_{d|4n}F_d(y)}{\prod_{d|2n}F_d(y)}=\prod_{\substack{d|4n\\ d\nmid 2n}}F_d(y)=\prod_{d|\frac{n}{e}}F_{4ed}(y).$$

The last equality follows from the fact that the required divisors must be multiples of $4e$ (lest they divide $2n$), and that any multiple of $4e$ must appear in the product. Hence such divisors are $4e$ times the odd divisors of n. Using the identity (24), we transform the RHS of (34) as follows

$$\begin{aligned}
\mathcal{M}'_{2n}(\lambda)\mathcal{V}_{n-k}(\lambda) &= \prod_{\substack{d|2n\\ d>2}}\Psi_d(\lambda)\prod_{d|\frac{n-k}{e}}\Psi_{4ed}(\lambda)\\
&= \prod_{\substack{d|2n\\ d>2}}F_d(y)y^{-\phi(d)/2}\prod_{d|\frac{n-k}{e}}F_{4ed}(y)y^{-\phi(4ed)/2}\\
&= \frac{1}{y^2-1}y^{-(2n-2)/2}(y^{2n}-1)y^{-(2n-2k)/2}(y^{2n-2k}+1)\\
&= \frac{1}{y^2-1}(y^{k+1}+y^{2n-k+1}-y^{-2n+k+1}-y^{-k+1})
\end{aligned}$$

which coincides with (36) with $t=2n$, as desired.

With reference to equations (29), we have established that $\mathcal{M}'_t$ divides D_t as well as each summand X_t in $\mathbb{Z}[\lambda]$, and hence this factor can be cancelled out. Furthermore, because $P_{t,k}=P_{t,t-k}$ —cf. (32)— the terms in (31) can be re-grouped to give the expressions (27) and (28). This completes the proof of the formula for x in (26). The formula for y is established by noting that $y_t=x_{t-1}$, and hence $\mathcal{Y}_t(\iota)=\mathcal{X}_t(\sigma^{-1}(\iota))$, where σ is the (left) shift map. □

The dependence of the periodic point formulae on the fundamental domain Ω appears only through the symbolic dynamics. Consequently, theorem 2 generalizes to any fundamental domain Ω of the lattice $\mathbb{Z}^2$, with the property that $G\Omega=\Omega$, because in this case there is no translation in the y-direction. We shall consider a specific family of this kind in section 6.

As a first application of the periodic point formulae (26), we characterize symmetric orbits by their code. Consider a periodic t-code ι, represented as the set of labelled vertices of a regular polygon with t vertices. We call the code *symmetric,* if the corresponding labelled polygon has a symmetry axis. We have the following

Proposition 3 *A periodic orbit is symmetric if and only if its code is symmetric.*

Proof. Assume first that the code is symmetric, and that the symmetry axis bisects one side of the polygon; call ι_0 and ι_{t-1} the adjacent vertices. Then $\iota_k=\iota_{t-k-1}$, for all k, from symmetry. With reference to equations (27) and (28) we let $\bar{\mathcal{V}}$ be equal to $\mathcal{V}$ or $\mathcal{W}$, depending whether $t = 2n$ or $t = 2n + 1$, respectively. We obtain

$$\begin{aligned} \mathcal{X}_t(\lambda, \sigma^{-1}(\iota)) &= \iota_{t-1}\bar{\mathcal{V}}_n(\lambda) + \sum_{k=1}^{n}(\iota_{k-1} + \iota_{t-k-1})\,\bar{\mathcal{V}}_{n-k}(\lambda) \\ &= \iota_0\bar{\mathcal{V}}_n(\lambda) + \sum_{k=1}^{n}(\iota_{t-k} + \iota_k)\,\bar{\mathcal{V}}_{n-k}(\lambda) = \mathcal{X}_t(\lambda, \iota). \end{aligned}$$

This shows that the orbit contains a point on Fix(G) (see equation (12). If the symmetry axis intersects two vertices, let ι_0 and ι_{t-2} be the vertices adjacent to one of them. Then $\iota_k = \iota_{t-k-2}$, from symmetry. The period is necessarily even, $t = 2n$, so we write

$$\begin{aligned} \mathcal{X}_{t-2}(\lambda, \sigma^{-2}(\iota)) &= \iota_{t-2}\mathcal{V}_n(\lambda) + \sum_{k=1}^{n}(\iota_{k-2} + \iota_{2n-k-2})\,\mathcal{V}_{n-k}(\lambda) \\ &= \iota_0\mathcal{V}_n(\lambda) + \sum_{k=1}^{n}(\iota_{2n-k} + \iota_k)\,\mathcal{V}_{n-k}(\lambda) = \mathcal{X}_t(\lambda, \iota). \end{aligned}$$

This shows that the orbit contains a point on Fix(H).

Conversely, assume that a t-cycle is symmetric. If the orbit has a point on Fix(G), then by shifting the code, if necessary, we may assume that $\mathcal{X}_t(\lambda, \iota) = \mathcal{X}_t(\lambda, \sigma^{-1}(\iota))$. We regard this as a polynomial identity in $\mathbb{Z}[\lambda]$. Because the polynomials $\mathcal{V}_k$ and $\mathcal{W}_k$ are monic of degree k, by matching the leading coefficient (degree n) we obtain $\iota_0 = \iota_{t-1}$. The next coefficient gives $\iota_1 + \iota_{t-1} = \iota_0 + \iota_{t-2}$, hence $\iota_1 = \iota_{t-2}$. This process is repeated until the symmetry of the code is established. If the t-cycle is symmetric but has no point on Fix(G), then it has a point of Fix(H). By shifting the code, if necessary, we arrive at $\mathcal{X}_t(\lambda, \iota) = \mathcal{X}_t(\lambda, \sigma^{-2}(\iota))$, and then proceed as above. □

Given a periodic code ι with valid symbols (recall that the alphabet depends on λ —see (5)), the periodic point formulae (26) represent an actual periodic point when $(x, y) \in \Omega$ for all cyclic permutations of ι (hence $(x, y) \in \Omega_\lambda$, defined in (17)). This condition defines a parametric interval(s) —or λ-interval— for which a given orbit exists. (This viewpoint was exploited in [6].) For fixed λ, the number of periodic orbits appears to grow algebraically with the period —see figure 2. Hence, among all periodic codes, the valid codes have zero density.

The endpoints $\lambda_\pm$ of such interval are *bifurcation points* for the cycle in question. At the bifurcation point the cycle may or may not exist, depending on whether the relevant coordinate x leaves the unit interval at $x = 0$ or $x = 1$, respectively. Accordingly, the λ-interval for which the orbit exists may be closed, open, or neither. From (26) it follows that the bifurcation points are roots of polynomials of the type

$$\mathcal{X}_t(\lambda, \sigma^k(\iota)) = 0 \qquad \mathcal{X}_t(\lambda, \sigma^l(\iota)) - \mathcal{M}_t(\lambda) = 0 \tag{37}$$

for a suitable k, l: in the first case the orbit exists at the bifurcation point, in the latter it doesn't. In equations (37), we are implicitly assuming that the polynomials in question are not identically zero if ι is not the zero code. This is indeed a case, by virtue of theorem 9, section 5: if $\mathcal{X}_t$ were zero, then any λ-interval which does not reduce to a point will include transcendental values.

Thus to each code ι we can associate a pair of irreducible polynomials $\beta_\pm(\lambda) \in \mathbb{Z}[\lambda]$, namely the minimal polynomials of $\lambda_\pm$, together with two flags which signal whether or not the orbit exists at the corresponding boundary value. The λ-interval of a code ι may not be unique: we specify it uniquely by providing a rational number with minimal denominator contained in it —the *dominant rational* λ_* of the interval. By virtue of theorem 9, the dominant rational is an interior point of the interval.

To estimate the size of a λ-interval, we proceed heuristically as follows. One would expect that every cycle of sufficiently long code ι will feature at least one rational function $x(\lambda, \sigma^k(\iota))$ having maximal degree, namely a function with maximal denominator $\mathcal{M}_t(\lambda)$. (Even though a drop of degree for some points of the cycle does take place, e.g., for the 4-cycle with code $\iota = (0, 0, 0, 1)$, we have verified this maximality property in all cases we have considered.) Under such circumstances, the dominant rational will be sandwiched between two adjacent roots of $\mathcal{M}_t$, leading to the estimates

$$2\cos(2\pi k/t) < \lambda_- < \lambda_* < \lambda_+ < 2\cos(2\pi(k-1)/t) \qquad (38)$$

for a suitable k. This would imply that the length of λ-intervals decreases at least as fast as the reciprocal of the period. In particular, only finitely many codes can survive any change of parameter.

We illustrate the above constructs with a detailed discussion of an example of an unstable cycle at an algebraic bifurcational point, which corresponds to an irrational rotation number. This example should also be seen in relation to theorem 9. We consider the symmetric 9-cycle with code $\iota = (1, 1, 1, 1, 1, 1, 2, 1, 1)$. Applying equation (26) we find that all points in the cycle have maximal denominator

$$\mathcal{M}_9(\lambda) = (\lambda - 2)(\lambda + 1)(\lambda^3 - 3\lambda + 1) = \Psi_1(\lambda)\Psi_3(\lambda)\Psi_9(\lambda)$$

and numerators $\mathcal{X}_k = \mathcal{X}(\lambda, \sigma^k(\iota))$ which we display as a product of irreducibles

$$\begin{aligned} \mathcal{X}_0 = \mathcal{X}_3 &= \lambda(\lambda^3 + \lambda^2 - 3\lambda - 1) \\ \mathcal{X}_1 = \mathcal{X}_2 &= (\lambda^2 + \lambda - 1)(\lambda^2 - 2) \\ \mathcal{X}_4 = \mathcal{X}_8 &= \lambda(\lambda^3 + \lambda^2 - 2\lambda - 3) \\ \mathcal{X}_5 = \mathcal{X}_7 &= \lambda^4 + 2\lambda^3 - 4\lambda^2 - 4\lambda + 2 \\ \mathcal{X}_6 &= 2(\lambda^2 + \lambda - 1)(\lambda^2 - \lambda - 1). \end{aligned} \qquad (39)$$

The dominant rational is $\lambda_* = -1/4$; hence, from maximality, we have a preliminary estimate of the λ-interval in terms of two adjacent roots of $\mathcal{M}_9$ —cf. (38)

$$-1 = 2\cos(2\pi 3/9) < \lambda_- < -\frac{1}{4} < \lambda_+ < 2\cos(2\pi 2/9) \approx 0.347$$

Because 0 is a root of $\mathcal{X}_0$, we improve the rightmost estimate to $\lambda_+ \leq 0$.

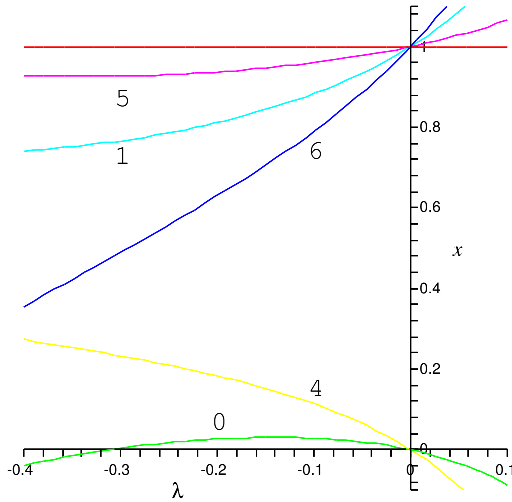

Figure 3: Graphs of the rational functions $x_k(\lambda)$ for the 9-cycle with code $\iota = (1,1,1,1,1,1,2,1,1)$. Due to symmetry, there are only 5 distinct rational functions —see (39)— corresponding to $k = 0,1,4,5,6$, which are the integers labelling the curves. The boundary line $x = 1$ is also plotted. Within the λ-interval of this code, these functions satisfy the inequalities $x_0 < x_4 < x_6 < x_1 < x_5$. We see that the left boundary of the λ-interval is determined by x_0, while any rational function x_k determines the right boundary.

From figure 3 we see that x_0 alone determines the λ-interval of the cycle, and since x_0 vanishes at the boundaries, the boundary polynomials are factors of $\mathcal{X}_0$

$$\beta_-(\lambda) = \lambda^3 + \lambda^2 - 3\lambda - 1 \qquad\qquad \beta_+(\lambda) = \lambda.$$

Thus the code ι determines a trivial field extension $\mathbb{Q}(\lambda_+) = \mathbb{Q}$, and a non-trivial one $\mathbb{Q}(\lambda_-)$, a cubic extension (with Galois group S_3). Because for some k, we have $x_k(\lambda_+) = 1$, we conclude that the right endpoint $\lambda_+ = 0$ does not belong to the interval; the left endpoint does, and is a root of $\beta_-(\lambda)$, namely

$$\lambda_- = -\frac{1}{3} - \frac{2}{3}\sqrt{10}\cos\left(\frac{2\pi - \arccos(10^{-3/2})}{3}\right) = -0.311108\ldots.$$

Thus at $\lambda = \lambda_-$ the map L has an unstable 9-cycle. We must now verify that the rotation number is irrational, that is, that the irreducible cubic polynomial β_- is not a Ψ-polynomial. The degree of Ψ_d is $\phi(d)/2$ (cf. equations (18) and remarks following it), and hence Ψ_d is cubic for $d \in \phi^{-1}(6) = \{7, 9, 14, 18\}$. For such values of d one verifies directly that $\beta_-(\lambda) \neq \Psi_d(\lambda)$, which shows that the rotation number of our cycle at $\lambda = \lambda_-$ is *irrational.* The representative point for this unstable cycle on the generator Γ_0 is

$$z = (x_1(\lambda_-), x_0(\lambda_-)) = (0.76271\ldots, 0).$$

We have established the existence of an unstable 9-cycle at a parameter value that corresponds to an irrational rotation number. This is the shortest cycle of L with the stated property. We have established this by automating the above procedure, and applying it to all periodic codes with up to 9 symbols, constructed in accordance with the alphabet (5). With the same line of reasoning, we get a general estimate

Theorem 4 *If λ is an algebraic number of degree d, then there are no unstable periodic points of period smaller than $2d$, apart from the fixed point at the origin.*

The above estimate gives the best uniform bound in d. Indeed, it can be verified that the periodic code $\iota = (0, 0, 0, 1)$ supports an unstable 4-cycle with initial condition $(x_0, x_{-1}) = (1/2, 0)$ at the quadratic parameter value $\lambda_- = \sqrt{2}$. Unlike in the previous example, the equation

$$\beta_-(\lambda) = \lambda^2 - 2 = \Psi_8(\lambda)$$

shows that the rotation number at bifurcation is *rational.*

Proof. The discontinuity set is generated by the segment $x = 0$, hence it suffices to consider periodic points of the form $(0, y)$, with $y \neq 0$. Assume for a moment that the rotation number θ satisfies the hypothesis of theorem 2. From equation (26), we see that such periodic points can exist only for parameters λ which are roots of the polynomial $\mathcal{X}_t(\lambda, \iota) \in \mathbb{Z}[\lambda]$, where ι is a non-zero code. Now, if $\mathcal{X}_t \neq 0$, the roots of $\mathcal{X}_t$ are algebraic numbers, and since the degree of $\mathcal{X}_t$ is at most $t/2$, if λ is algebraic of degree $t < 2d$, then we have $\mathcal{X}_t(\lambda) \neq 0$. The periodic point formulae hold if the rotation number θ is irrational. If θ is rational, then the possible values of its denominator b are determined from the degree d by the condition $b \in \phi^{-1}(2d)$. For $x > 1$, we have $\phi^{-1}(x) > x$ (because $\phi(x) < x$), and hence $b > 2d$. Therefore, as long as $t \leq 2d$, we have $t < b$, and, in particular, the denominator b does not divide the period t. So, for the range of periods considered, theorem 2 applies to the rational case as well. □

We remark that the argument used in the proof of the theorem is not sufficient to establish the absence of unstable periodic orbits for *transcendental* parameter values, because for $\lambda > 1$ a non-zero code ι could conceivably correspond to the zero polynomial $\mathcal{X}$. We shall prove this —and more— in section 5, using non-archimedean methods.

We conclude this section by deriving an estimate of the complexity of periodic points in the case in which $\lambda = a/b$ is a rational number. We know that in the present case all periodic points are rational; with reference to equation (26), we define the sequence

$$M_t = b^{\lfloor (t+2)/2 \rfloor} \left| \mathcal{M}_t \left(\frac{a}{b} \right) \right| \qquad t = 1, 2, \ldots. \tag{40}$$

Because $\mathcal{M}_t(\lambda)$ is a polynomial with integer coefficients of degree $\lfloor (t+2)/2 \rfloor$, and $\deg(\mathcal{X}_t) < \deg(\mathcal{M}_t)$, the M_t are integers, and represent an upper bound on the denominator of any rational point (if any) which is periodic of period t. For the sequence M_t, we have the following exponential bound

Proposition 5 *Let $\lambda = a/b$. We have the following estimates*

$$M_t \leq 2\, b^{t/2} \times \begin{cases} \sqrt{2b-a} & t \text{ odd} \\ \sqrt{4b^2-a^2} & t \text{ even.} \end{cases}$$

Proof. These derive at once from (40) and the inequalities

$$|\mathcal{M}_t(x)| \leq 2\sqrt{2-x} \qquad t \text{ odd}; \qquad\qquad |\mathcal{M}_t(x)| \leq 2\sqrt{4-x^2} \qquad t \text{ even.}$$

We establish the inequality for even t. Let $t = 2n$; using (18) and (35), with $x = y + y^{-1}$ and $|y| = 1$, we find

$$\frac{|\mathcal{M}_{2n}(x)|}{\sqrt{4-x^2}} = |\prod_{d|2n} F_d(y) y^{-\phi(d)/2}| = |y^{2n} - 1| \leq 2.$$

The proof for odd t is very similar and we omit the details. □

4 Pseudo-hyperbolic points

For a set of full measure of parameter values, the map L does not possess unstable periodic orbits (theorem 9, section 5). We now introduce the surrogate concept of *pseudo-hyperbolic points;* these points are in fact neither periodic nor hyperbolic, but their similarities with unstable periodic points will justify the terminology. Here we develop definitions and basic formulae; we shall return to the study of these points in section 5, in regard to their non-archimedean properties.

A point z of the discontinuity set is said to be *pseudo-hyperbolic* if there exist integers $t_\pm$ such that

$$z_+ := L^{t_+}(z) \in \partial\Omega \qquad\qquad z_- := L^{-t_-}(z) \in \partial\Omega, \qquad t_\pm \geq 0$$

where $\partial\Omega$ —the boundary of the atoms of L— was defined in (6). The integers $t_\pm$ are chosen so as to be minimal, so that $z_\pm$ are uniquely determined by z. (Note that, by construction, one of the integers t_-, t_+ always exists, the issue being the existence of the other.) All points sharing the same boundary points $z_\pm$ form a *pseudo-hyperbolic sequence;* the *length* of the sequence is the number of its elements. We call z_- and z_+ the left and right boundary points of the sequence, respectively; thus time flows from left to right. A pseudo-hyperbolic orbit is an orbit containing a pseudo-hyperbolic sequence.

There is a certain degree of flexibility in the above definition: on the one hand, the boundary set $\partial\Omega$ could be replaced by that of some iterate of the map L, with greater complexity; on the other, it could be replaced by a generator of Γ, with minimal complexity. For our purpose, the most convenient choice is to identify $\partial\Omega$ with $\Gamma_0 \cup G\Gamma_0$, where the generator Γ_0 and the reversor G were defined in section 2. This choice is motivated by the fact that, as maps of the torus, L and L^{-1} are discontinuous on $G\Gamma_0 = L^{-1}(\Gamma_0)$, and on Γ_0, respectively.

Clearly, all unstable periodic orbits are pseudo-hyperbolic, and they contain infinitely many pseudo-hyperbolic sequences. The simplest example is the fixed point $z = (0, 0)$ at the origin. A less trivial example is the 4-cycle mentioned after the statement of theorem 4 ($\lambda = \sqrt{2}$, $z_0 = (1/2, 0)$), which corresponds to a *rational* rotation number. However, these are not typical examples: one can easily see that for an irrational rotation number, all pre-images of $G\Gamma_0$ and images of Γ_0 have distinct orientations. Therefore, apart from the origin,[8] all pseudo-hyperbolic points correspond to *transversal* intersections of two segments γ and $\gamma\prime$ of Γ. (We say that two segments γ and $\gamma\prime$ intersect transversally if $\gamma \cap \gamma\prime$ is a single point, which is different from any of the endpoints of the two segments.) Here the intersecting segments play the role of 'separatrices' in smooth systems, hence the terminology.

As we did for periodic orbits, we distinguish between symmetric and non-symmetric sequences; however, now reversibility is intended as a property of an individual sequence, not of an orbit; that is, a symmetric orbit could conceivably have non-symmetric sequences. To construct the symmetric pseudo-hyperbolic sequences of a general reversible map $L = HG$, one proceeds as follows. Let the discontinuity set of L have generator Γ_0 and boundary set $\partial\Omega = \Gamma_0 \cup G\Gamma_0$, and let σ be a pseudo-hyperbolic sequence with left boundary point z_-. From symmetry, $Gz_- \in \sigma$, hence $Gz_- = L^s z_-$, for some $s \geq 0$, chosen so as to be minimal. Using (11) we find that $GL^n z_- = L^{s-n} z_-$ for all $n \in \mathbb{Z}$, and $GL^n z_- \in \sigma$ for $n = 0, 1, \ldots, s$. By construction, Gz_- belongs to $\partial\Omega$ and s is minimal, so the right boundary point z_+ of σ is Gz_-, and σ has length $t = s + 1$. If $s = 2n$ is even, we obtain $GL^n z_- = L^n z_-$, that is, $L^n z_- \in \mathrm{Fix}(G)$. Therefore the pseudo-hyperbolic sequences of odd length $t = 2n + 1$ are determined by constructing the finite set

$$\Gamma_n \cap \mathrm{Fix}(G) \qquad \Gamma_n = L^n(\Gamma_0) \tag{41}$$

and then removing from it the points $\Gamma_k \cap \mathrm{Fix}(G)$, $k = 1, 2, \ldots, n-1$. (However, the latter would correspond to periodic orbits on Γ, so this may be unnecessary, in accordance with theorem B.) Note the analogy with the algorithm for finding symmetric periodic orbits —see remarks following equation (12). If $s = 2n - 1$ is odd, we obtain

$$GL^{n-1} z_- = L^n z_- \qquad \Longleftrightarrow \qquad L^n z_- = HGL^{n-1} z_- = HL^n z_- \tag{42}$$

that is, $L^n z_- \in \mathrm{Fix}(H)$. This time we find the sequences of even length $t = 2n$ by intersecting Γ_n with $\mathrm{Fix}(H)$.

We specialize the above to our map (2), providing formulae for the boundary points of a hyperbolic sequence of L with a given code ι. These formulae —the analogue of theorem 2 for periodic points— represent boundary points as rational functions in λ, expressed in terms of the Ψ-polynomials, with denominator factored into irreducibles. We need a new family of polynomials, defined for even indices only

$$\mathcal{M}''_{2t}(x) = \frac{\mathcal{M}'_{2t}(x)}{\mathcal{M}'_t(x)} = \prod_{\substack{d|2t \\ d>2, d\nmid t}} \Psi_d(x) \qquad t = 1, 2, \ldots. \tag{43}$$

[8]In general, one must exclude the finitely many sequences whose boundary points $z_\pm$ are end-points of the segments comprising $\partial\Omega$.

Theorem 6 *Let $z_0 = (x, 0)$ and $z_{t-1} = (0, y)$ be the left and right boundary points, respectively, of a pseudo-hyperbolic sequence of length t and code $\iota = (\iota_0, \iota_1, \dots)$. Let the rotation number θ be either irrational, or rational with denominator not divisible by $2t$. Then*

$$x = -\frac{1}{\mathcal{M}'_{2t}} \sum_{k=0}^{t-2} \iota_{t-2-k} S_k \qquad y = -\frac{1}{\mathcal{M}'_{2t}} \sum_{k=0}^{t-2} \iota_k S_k. \tag{44}$$

If the orbit is symmetrical, then

$$x = y = -\frac{1}{\mathcal{M}''_{2t}} \times \begin{cases} (R_{n-1} - R_{n-2}) & t = 2n+1 \\ (R_{n-1} - R_{n-3}) & t = 2n. \end{cases} \tag{45}$$

Of note is the fact that a *finite* portion of the code ι suffices to determine uniquely a non-periodic orbit.

Proof. From lemma 1, the equation $x_{t-1} = 0$ gives $x = -R_{t-2}/S_{t-1}$. From (25), we have $S_{t-1} = \mathcal{M}'_{2t}$. For the right boundary point, we find, from the same lemma

$$\begin{aligned} y = x_{t-2} &= xS_{t-2} + R_{t-3} = -\frac{1}{S_{t-1}}(R_{t-2}S_{t-2} - S_{t-1}R_{t-3}) \\ &= -\frac{1}{S_{t-1}}\left[\sum_{k=0}^{t-2} \iota_k S_{t-2-k} S_{t-2} - \sum_{k=0}^{t-3} \iota_k S_{t-3-k} S_{t-1}\right] \\ &= -\frac{1}{S_{t-1}}\left[\iota_{t-2} S_0 S_{t-2} + \sum_{k=0}^{t-3} \iota_k (S_{t-2} S_{t-1-(k+1)} - S_{t-1} S_{t-2-(k+1)})\right] \\ &= -\frac{1}{S_{t-1}}\left[\iota_{t-2} S_{t-2} + \sum_{k=0}^{t-3} \iota_k S_k\right] \\ &= -\frac{1}{S_{t-1}} \sum_{k=0}^{t-2} \iota_k S_k \end{aligned}$$

where we have used the identity (15). Thus if θ is irrational, or rational with denominator not dividing the $2t$, then the denominator of x and y is non-zero. In the symmetric case, the coordinate x is computed from (41), (12), and lemma 1, by solving the equations $xS_n + R_{n-1} = xS_{n-1} + R_{n-2}$ if $t = 2n+1$ is odd, and the equation $xS_n + R_{n-1} = xS_{n-2} + R_{n-3}$ if $t = 2n$ is even.

It remains to factor the denominators. We begin with the case $t = 2n$. The identity (15) with $k = 2n$ and $t = n-1$, together with (16) give

$$S_{n-2}S_{-(n+1)} - S_{n-1}S_{-(n+2)} = -S_{n-2}S_{n-1} + S_{n-1}S_n = S_{2n-1}.$$

This shows that $S_n - S_{n-2} = S_{2n-1}/S_{n-1} = \mathcal{M}'_{2t}/\mathcal{M}'_t$, as desired. For odd $t = 2n+1$, we proceed as in the proof of formula (36). Using the same notation, we find, on the one hand

$$S_n - S_{n-1} = \mathcal{M}'_{2n+2} - \mathcal{M}'_{2n} = \frac{\prod_{\substack{d|2n+2 \\ d>2}} \Psi_d}{\prod_{\substack{d|2n \\ d>2}} \Psi_d}$$

$$= \frac{1}{y^2-1}\left[y^{n+2} - y^{-n} - y^{-n+1}(y^{2n}-1)\right] = \frac{y^{n+1}+y^{-n}}{y+1}$$

and on the other

$$\frac{\mathcal{M}'_{4n+2}}{\mathcal{M}'_{2n+1}} = \frac{\prod_{\substack{d|4n+2\\ d>2}} \Psi_d}{\prod_{\substack{d|2n+1\\ d>2}} \Psi_d} = \frac{(y-1)\,y^{-2n}(y^{4n+2}-1)}{(y^2-1)\,y^{-n}(y^{2n+1}-1)} = \frac{y^{n+1}+y^{-n}}{y+1},$$

as desired. □

An example of growth of the number of symmetric and non-symmetric pseudo-hyperbolic sequences with their length is displayed in figure 4. Unlike for periodic orbits, pseudo-

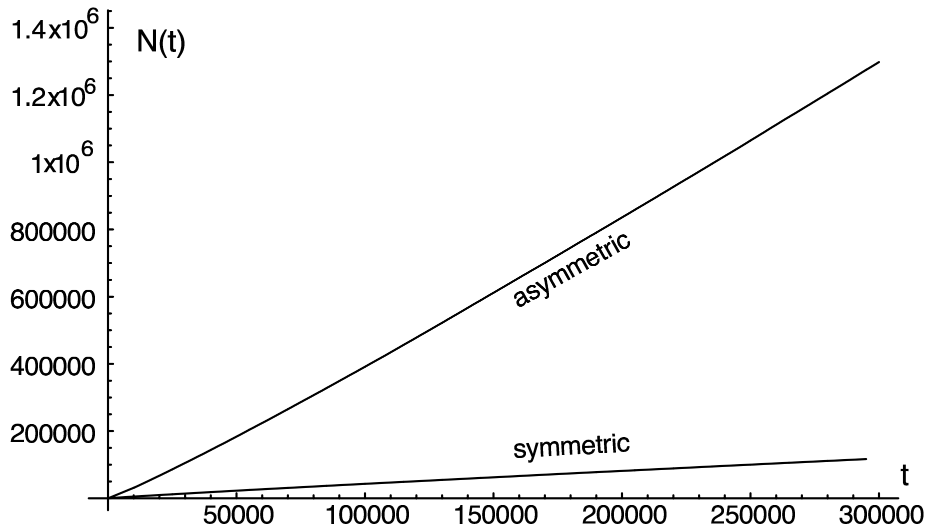

Figure 4: Number $N(t)$ of pseudo-hyperbolic sequences of length not exceeding t, at $\lambda = 1/2$, for the symmetric and asymmetric cases. Over the measured range, the growth is approximately algebraic, namely $t^{0.92}$ (symmetric) and $t^{1.09}$ (asymmetric).

hyperbolic sequences which are non-symmetric are common.

Remark. The left boundary point $z = (x, 0) \in \Gamma_0$ of all pseudo-hyperbolic sequences form a distinguished subset of Γ_0. Each of them is determined by the value of a rational function in λ, given by equation (44). Therefore an orbit that exists for a specific value of λ, will also exist within a parametric interval. As the parameter is varied, a pseudo-hyperbolic orbit can collide with adjacent orbits, or disappear when one of the points in the orbit leaves Ω. For instance, the unstable 9-cycle described in the previous section results from the concatenation of two symmetric pseudo-hyperbolic sequences, the 3-sequence with code $(1, 1, \ldots)$, and the 6-sequence with code $(1, 1, 2, 1, 1, \ldots)$. They collide at $\lambda = \lambda_-$.

5 Non-archimedean properties

In this section we develop non-archimedean methods, and apply them to the study of periodic and pseudo-hyperbolic points. For completeness, we provide a succinct account of the main concepts; background material can be found, for instance, in [20] or [22].

Fix a prime p. The *p-adic value* $\nu_p(m)$ of an integer m is defined to be the largest k such that p^k divides m, with $\nu_p(0) = \infty$. Letting $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$, we extend the definition of ν_p to the rationals. The function ν_p —called a *valuation*— has the following properties

$$\begin{array}{lll} (i) & \nu_p(rs) = \nu_p(r) + \nu_p(s) & \\ (ii) & \nu_p(r+s) \geq \min(\nu_p(r), \nu_p(s)) & r, s \in \mathbb{Q} \\ (iii) & \nu_p(r) \neq \nu_p(s) \Rightarrow \nu_p(r+s) = \min(\nu_p(r), \nu_p(s)) & \end{array} \tag{46}$$

The inequality (ii), together with (iii), is referred to as the *ultrametric inequality.* The set of rationals with non-negative p-adic value is a ring, denoted by $\mathbb{Z}_p$; they are the rational p-adic integers. For $0 \neq r \in \mathbb{Q}$, we write

$$r = \mathcal{U}(r)\, p^{\nu_p(r)} \qquad \nu_p(\mathcal{U}(r)) = 0. \tag{47}$$

The quantity $\mathcal{U}(r)$ is a *unit,* namely an invertible element of $\mathbb{Z}_p$. For instance, if $\lambda = a/p^n$, the maximal denominator M_t of a t-cycle (equation (40)), is expressed concisely as $M_t = \mathcal{U}(\mathcal{M}_t)$.

The function ν_p is related to the notion of 'p-adic absolute value' $|\cdot|_p$ via the equation

$$|r|_p = p^{-\nu_p(r)} \qquad r \in \mathbb{Q}. \tag{48}$$

We see that $\mathbb{Z}_p$ is just the p-adic unit disc $|r|_p \leq 1$, while the units constitute the p-adic unit circle $|r|_p = 1$. The values assumed by $|\cdot|_p$ are discrete; since $\nu_p(\mathbb{Q}) = \mathbb{Z}$, these values are p^n, $n \in \mathbb{Z}$. The p-adic absolute value induces a metric on the rationals, with distance $d(r, s) = |r - s|_p$, since the ultrametric inequality implies the triangle inequality.

If b is an integer, not necessarily prime, we define

$$\nu_b(r) = \sum_{p|b} \nu_p(r).$$

This function satisfies the equality (46) (i), but *not* the ultrametric inequality (ii), (iii), (e.g., $\nu_6(3+4) = \nu_6(7) = 0$, whereas $(\nu_6(3) = 1,\ \nu_6(4) = 2)$). Even though the pseudo-valuation ν_b does not induce a metric, it will prove useful in dealing with the case of rational parameter values.

An analogous construct can be developed for the rational function field $\mathbb{Q}(\lambda)$. Here λ is an indeterminate, that is, a transcendental element over $\mathbb{Q}$. For a polynomial $f(\lambda)$ with rational coefficients we let $\nu_\infty(f) = -\deg(f)$, with $\nu_\infty(0) = \infty$. Then we extend the definition of ν_∞ to rational functions $r(\lambda) = f(\lambda)/g(\lambda)$ via $\nu_\infty(r) = \nu_\infty(f) - \nu_\infty(g)$. It can be shown that ν_∞ satisfies the same properties (46) as ν_p. As we did with ν_p, from the valuation ν_∞ we obtain a non-archimedean absolute value $|\cdot|_\infty$, by letting $|r|_\infty = e^{-\nu_\infty(r)}$. There are other valuations for $\mathbb{Q}(\lambda)$, but ν_∞ will suffice for our purpose.

We generalize the above definitions to two dimensions by letting $\nu(z) = \min(\nu(x), \nu(y))$, where $z = (x, y)$. In terms of the absolute value, this is equivalent to $|z|_p = \max(|x|_p, |y|_p)$, which defines a metric on $\mathbb{Q}^2$. Likewise, we let $\mathcal{U}(z) = (\mathcal{U}(x), \mathcal{U}(y))$.[9] If $h \leq k$ we speak of a *type I point,* and of a *type II point* if $h > k$. The same generalizations apply also to the valuation ν_∞, and to the pseudo-valuation ν_b.

We now derive a valuation-theoretic characterization of periodic and pseudo-hyperbolic orbits. For rational values of λ, we employ the following notation

$$\lambda = \frac{a}{b} \in \mathbb{Q} \qquad b = \prod_k p_k^{\alpha_k} \qquad n = \sum_k \alpha_k \qquad \gcd(a, b) = 1 \tag{49}$$

where the p_k are distinct prime numbers and the exponents α_k are positive. In the statement of the next two theorems, we unify the notation as follows: if λ is transcendental, we let $\nu = \nu_\infty$ and $n = 1$; if λ is rational, we let $\nu = \nu_b$, with b and n as in (49). We cannot however unify the proofs in the two cases, because for rational λ we cannot rely on the ultrametric inequality (46).

Theorem 7 *Let λ be transcendental or rational, and let z be a periodic point of the map L. Then $\nu(z) \geq n$. For $z \neq (0, 0)$, this estimate can be refined as follows:*

(i) If $\lambda < 0$ and λ is transcendental, or λ is rational with denominator coprime to 6, then $\nu(z) = n$

(ii) If $\lambda > 0$, then every periodic orbit contains a point z such that $\nu(z) = n$.

Proof. The first statement is trivially true for $z = (0, 0)$, since $\nu(0) = \infty$. To compute values, we use the formulae (29) and (30). Because the polynomial $S_t(\lambda)$ is monic of degree t, so is $-D_t$, and hence $\nu(D_t) = -nt$. This follows from the ultrametric inequality if λ is transcendental; for λ rational, we verify that $b^t D_t(a/b) \equiv -a^t \pmod{b}$, and since a is coprime to b, we obtain, using (46 *i*)

$$\nu_b(-b^t D_t(a/b)) = t\nu_b(b) + \nu_b(D_t(a/b)) = tn + \nu_b(D_t(a/b)) = 0$$

as desired. Likewise, from equation (31) we have that $\nu(X_t) \geq -n(t-1)$, and therefore $\nu(x) = \nu(X_t) - \nu(D_t) \geq n$. To compute $\nu(y)$ is suffices to shift the code, and so we have the same estimate.

To refine the above estimate, assume $z \neq (0, 0)$, so that ι is not the zero code. With reference to (5), we note that for $\lambda < 0$ all symbols in the alphabet are non-zero. This suffices to establish the equality $\nu(z) = n$ is λ is transcendental, and also if λ is rational,

[9] We use the notation ν, $|\cdot|_p$, $\mathcal{U}$ interchangeably for $\mathbb{Q}$ and $\mathbb{Q}^2$.

long as every symbol ι_k is coprime to b. Because the only prime divisors of the alphabet are 2 and 3, it suffices to require that $\gcd(b,6)=1$. (If the alphabet contains some symbol ι_k not coprime to b, and if this holds for all non-zero symbols of an orbit, we would have the strict inequality $\nu(z)>n$.)

If $\lambda>0$, the alphabet has no prime divisors. However, the existence of zero symbols means that $\nu(X_t)$ assumes its minimal value $-n(t-1)$ precisely if the corresponding symbol ι_0 is non-zero; this can always be achieved with a shift, since ι is not the zero code, in which case $\nu(X_{t-1})\geq\nu(X_t)$, and hence $\nu(z)$ attains its minimum value $\nu(z)=n$. □

Theorem 8 *Let λ be transcendental or rational, and let σ be a pseudo-hyperbolic sequence of the map L, which is not the fixed point at the origin. Let z_- and z_+ be the left and right boundary points of σ, respectively. Then*

(i) $z\in\sigma\Rightarrow\nu(z)\geq n$

(ii) $\nu(L^t(z_+))=\nu(L^{-t}(z_-))=(1-t)n \qquad t\geq 0.$

Furthermore, all images of z_+ are of type I, and all pre-images of z_- are of type II.

Proof. Let σ be a pseudo-hyperbolic t-sequence with boundary points $z_-=z_0=(x,0)$ and $z_+=z_{t-1}=(0,y)$.

Let λ be a transcendental number, which we regard as an indeterminate. From the fact that $\nu_\infty(S_t(\lambda))=-t$, together with equations (44) and (46*i*), we have $\nu_\infty(x)\geq -\deg(S_{t-2}(\lambda))+\deg(S_{t-1}(\lambda))=1$, and, similarly, $\nu_\infty(y)\geq 1$. Thus the value of both boundary points is not smaller than 1; we have to show that the same property holds for any element in the sequence. Suppose that σ has more than one element, that $z=(x_k,x_{k-1})$ is a pseudo-hyperbolic point in σ, and that k is the smallest positive integer such that $\nu_\infty(x_k)<1$. Then $\nu_\infty(x_k)<\nu_\infty(x_{k-1})$. Letting $\nu_\infty(x_k)=-m$ and $\nu_\infty(x_{k-1})=1+n$, with $m,n\geq 0$, from lemma 1, we find

$$\nu(x_{k+s})=\nu(x_kS_s-x_{k-1}S_{s-1}+R_{s-1}) \qquad s\geq 0$$

with

$$\nu_\infty(x_kS_s)=-m-s \qquad \nu_\infty(x_{k-1}S_{s-1})=2+n-s \qquad \nu_\infty(R_{s-1})\geq 1-s.$$

Applying (46*iii*) gives $\nu(x_{k+s})=-m-s\leq 0$, but for $k+s=t-1$ we know that $\nu_\infty(x_{k+s})=\nu(y)\geq 1$, a contradiction. Thus $\nu_\infty(z)\geq 1$ for all point z of the sequence σ.

Next we examine the points outside the sequence. Firstly, $z_t=(1-y,0)$, and hence $\nu_\infty(z_t)=\nu_\infty(1-y)=\min(\nu_\infty(1),\nu_\infty(y))=0$. Secondly, $\nu_\infty(R_t)\geq -t$, and from lemma 1 we have that

$$\nu_\infty(x_{\tau+t})=\nu_\infty((1-y)S_\tau(\lambda)-R_{\tau-1}(\lambda))=-\tau, \qquad \tau\geq 0.$$

Because $L^\tau(z_+) = (x_{\tau+t}, x_{\tau-1+t})$, the above equality also shows that $\nu_\infty(L^\tau(z_+)) = -\tau$, and that such point is of type I. To prove the symmetric statement for $L^{-\tau}(z_-)$ we exploit the time-reversal symmetry, and the fact that the coordinates of all the points which do not belong to σ have distinct value, so that the reversor G turns a type I point into type II and vice-versa.

Let now λ be a rational number. Using the notation (49), we write

$$S_s\left(\frac{a}{b}\right) = \frac{A_s}{b^s} \qquad A_s \equiv a^s \pmod b, \qquad s \geq 0. \tag{50}$$

Similarly, if j is the smallest index in the range $0 \leq j \leq s$ for which $\iota_{t-j} \neq 0$, we have

$$R_s\left(\frac{a}{b}\right) = \frac{A'_j}{b^j} \qquad A'_j \equiv \iota_{s-j} a^j \pmod b, \qquad s \geq 0. \tag{51}$$

This means

$$\frac{R_{t-2}}{S_{t-1}} = b^{t-1-j}\frac{A'_j}{A_{t-1}},$$

and because $j \leq t-2$ and A_{t-1} is coprime to b, it follows that the numerator of x is divisible by b, and, in particular $\nu_b(x) \geq (t-1-j)n \geq n$. Likewise, we find that b divides the numerator of y, with the same estimate: $\nu_b(y) \geq (t-1-j)n \geq n$. As above, we now to show that such estimate is valid also for the interior points of the sequence. Assuming that k is the smallest positive index such that $\nu_b(x_k) < n$, and considering that $\nu_b(x_{k-1}) \geq n$, we have, from lemma 1 $x_\tau = S_\tau x_k - S_{\tau-1}x_{k-1} + R_{\tau-1}$. Using the notation (50) and (51) we find

$$x_{k+s} = b^{-s}\left(A_s x_k - bA_{s-1}x_{k-1} + b^{s-j}A'_j\right) \qquad s \geq 0, \quad 0 \leq j \leq s-1.$$

Because, by assumption, the numerator of x_k is not divisible by b, we obtain the estimate $\nu_b(x_{k+s}) < (1-s)n \leq 0$. As for the transcendental case, letting $k+s = t-1$ leads to a contradiction.

From the fact that $\nu_b(y) \geq n$, it follows that $\nu_p(1-y) = 0$ for each prime divisor p_k of b, and hence $\nu_b(z_t) = 0$. From lemma 1, we find that for all $\tau \geq 0$, $x_\tau = S_\tau(1-y) + R_{\tau-1}$. We have

$$x_\tau = b^{-t}(A_\tau(1-y) + b^{\tau-1-j}A'_j)$$

and since A_τ is coprime to b, and A'_j is an integer, we conclude that $\nu_b(L^\tau(z_t)) = -\tau n$, and that all these points are of type I. As above, the statement concerning pre-images of z_- is dealt with using reversibility. □

Under the assumptions of the above theorem, it is plain that an orbit can have at most one pseudo-hyperbolic sequence. Furthermore, considering that an unstable periodic orbit features infinitely many recurrences to the generator Γ_0, and hence contains infinitely many pseudo-hyperbolic sequences, we obtain effortlessly the main structure theorem of this paper

Theorem 9 *If λ is transcendental or rational, then every orbit of L contains at most one pseudo-hyperbolic sequence; in particular, L has no unstable periodic orbits. (The fixed point at the origin is excluded from consideration.)*

It is plain that a symmetric pseudo-hyperbolic sequence belongs to a symmetric orbit. An immediate consequence of theorem 9 is that, for transcendental or rational parameter values the converse is also true, because any symmetric orbit containing one non-symmetric sequences, must necessarily have another one, namely its image under G.

We may use formulae (44) to determine the parameters supporting unstable periodic cycles, with a given number of recurrences to the generator Γ_0. For instance, if an unstable periodic orbit closes up at its first recurrence, then $L(z_+) = z_-$. In equation (44), this translates into $y = 1 - x$, and hence

$$-\frac{1}{S_{t-1}}\sum_{k=0}^{t-2}\iota_{t-2-k}S_k = 1 + \frac{1}{S_{t-1}}\sum_{k=0}^{t-2}\iota_k S_k \quad \Longrightarrow \quad S_{t-1} + \sum_{k=0}^{t-2}(\iota_k + \iota_{t-2-k})S_k = 0.$$

Generically, orbits of this type will be asymmetric. The polynomial

$$f_t(\lambda) = S_{t-1}(\lambda) + \sum_{k=0}^{t-2}(\iota_k + \iota_{t-2-k})S_k(\lambda) \tag{52}$$

is a monic polynomial over $\mathbb{Z}$, of degree $t-1$, whose roots are the parameter values for which a periodic orbits of that code may exist. (This makes it plain that such parameters are algebraic, and indeed not rational, because $f_t(\lambda)$ is monic.) Higher-order recurrences to Γ_0 are also possible. For instance, an unstable symmetric cycle generically features *two* recurrences, and hence it may be constructed by joining together two symmetric pseudo-hyperbolic sequences —see example at the end of section 4. Thus while pseudo-hyperbolic sequences —like stable periodic orbits— exist in an interval of parameters, unstable periodic orbits can only exist at isolated algebraic parameter values.

Theorem 8 states that, using a non-archimedean metric, a pseudo-hyperbolic sequence is contained in a small disk near the origin, whereas, outside the sequence, orbits escape to infinity in a regular fashion, in both time directions. Experiments show that if we choose a rational initial condition z 'at random', this behaviour is typical, even for initial conditions very close to the origin. We now prove that this is indeed the case. For the sake of simplicity, we restrict to the case $\lambda = a/p^n$, p a prime.

Lemma 10 *Let $\lambda = a/p^n$, and let z be a rational point which is not periodic. Then, for all sufficiently large $|t|$*

$$\nu(L^{t+\mathrm{sign}(t)}(z)) - \nu(L^t(z)) = -n.$$

Proof. Assume that $z_0 = (x_0, x_{-1})$ is a rational point which is not periodic. With the notation of lemma 1, we let

$$z_t = (x_t, x_{t-1}), \qquad \mathcal{U}(x_t) = \frac{a_t}{b_t} \qquad t \geq 1$$

where $\mathcal{U}$ was defined in (47). Then b_t divides $d = \mathrm{lcm}(b_0, b_{-1})$ for all t, and since $x_t \in [0,1)$, we have that $\nu(x_t)$ is bounded from above (by $\lfloor \log_{p^n}(d) \rfloor$). Now, if $\nu(x_t)$ were also bounded from below, the resulting set of points would be finite. So every non-periodic orbit must contain a point $z = (x, y)$ with negative value. If such point is of type I, then, from lemma 1 we see that its image will also be of type I, so that the value of z_t will decrease by n at every iteration, which would prove our result in the forward time direction. We now show that such type I point exists. Let $z = (x, y)$ be a type II point with negative value k. If $x = 0$, then $L(z) = (1 - y, 0)$ is a point of type I with negative value, so we can assume $xy \neq 0$. Then, for some $m \geq 1$ we have $k+m = \nu(x) > \nu(y) = k$, and there exist non-zero rationals r, s such that

$$x' = p^k(rp^{m-n} - s) + \iota(x, y), \qquad \nu(r) = \nu(s) = 0.$$

One verifies that for $k < 0$ we have

$$\begin{array}{lcl} m \geq n & \Longrightarrow & \nu(x') \geq k \\ m < n & \Longrightarrow & \nu(x') = k + m - n \end{array}$$

which shows that if $n = 1$, z cannot map into a type II point of lower value. If $n > 1$, this does happen if $m < n$; however, the image of such point is a type I point of negative p-adic value, whose existence is therefore established.

We have proved that, for all sufficiently large t, $\nu(z_{t+1}) - \nu(z_t) = -n$. The analogous statement for $t \to -\infty$ now follows from reversibility via (11), by repeating the above reasoning for the point $z = (y, x)$. □

Putting together theorem 8 and the above lemma, we obtain a complement to theorem 7: for rational points, p-adic boundedness is *sufficient* for periodicity. We shall express this fact as an asymptotic property of rational points, with the aid of a complexity function. For rational $x = a/b$, we first let

$$H(x) = H\left(\frac{a}{b}\right) = \max\{|a|_p, |b|_p\}.$$

Then we extend the definition to points $z = (x, y) \in \mathbb{Q}^2$ as $H(z) = \max\{H(x), H(y)\}$, and finally define the height h by taking the logarithm to the base p

$$h(z) = \log_p H(z).$$

Next we define the *height* [10]

$$\hat{h}(z) = \lim_{t\to\infty} \frac{1}{nt} h(L^t(z)) \tag{53}$$

provided that the limit exists.

Theorem 11 *Let $\lambda = a/p^n$, and let z be a rational point. Then*

$$\hat{h}(z) = \begin{cases} 0 & z \textit{ is periodic} \\ 1 & z \textit{ is not periodic.} \end{cases}$$

[10] This term is borrowed from arithmetic geometry (see, e.g., [36, chapter III]).

Proof. If an orbit is periodic, then h is bounded and hence $\hat{h}$ is zero. Conversely, assume that z is non-periodic. From lemma 10 there exists an integer t^* such that, for $t > t^*$, the denominator of $L^t(z)$ (rather, of its first coordinate) is equal to $p^{n(t-t^*)}b$, for some integer b coprime to p. Therefore

$$\hat{h}(z) = \lim_{t\to\infty} \frac{1}{nt}\left[n(t-t^*) + \log_p(b)\right] = 1,$$

as desired. □

We conclude this section by considering the question of p-adic embedding. We have seen that, p-adically speaking, rational points behaves as though they were affected by unstable equilibria. This is indeed the case. The eigenvalues of the matrix S are roots of the polynomial $f(x) = x^2 - \lambda x + 1$. For $\lambda = a/p^n$, we find

$$\lambda_{\pm} = \frac{1}{2p^n}(a \pm \sqrt{a^2 - 4p^{2n}}).$$

The discriminant of $f(x)$ is a non-zero square modulo p: $a^2 - 4p^{2n} \equiv a^2 \pmod{p}$; from Hensel's lemma [22, p. 169] it follows that $f(x)$ has two distinct roots in the p-adic field $\mathbb{Q}_p$ —the completion of $\mathbb{Q}$ with respect to the metric $|\cdot|_p$. Therefore it is possible to construct p-adic rational approximations to $\lambda_{\pm}$. Using the binomial series[11]

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{2^3}x^4 + \frac{1}{2^4}O(x^6)$$

which converges in a suitable neighbourhood of the origin [22, p. 265], we obtain (recall that p is 'small')

$$\lambda_{\mp} = \frac{1}{p^n}\left[a\,\frac{1 \mp \mathrm{sign}(a)}{2} \pm \frac{\mathrm{sign}(a)}{a}p^{2n} + O(p^{4n})\right] \tag{54}$$

which implies $|\lambda_+|_p > 1$ and $|\lambda_-|_p < 1$ if $a > 0$, and $|\lambda_-|_p > 1$ and $|\lambda_+|_p < 1$ if $a < 0$. Thus the fixed point at the origin, as well as all periodic points are *hyperbolic.*

The natural embedding $\mathcal{E} : \mathbb{Q} \to \mathbb{Q}_p$ may be used to embed $\Omega_{\mathbb{Q}}$ into $\mathbb{Q}_p^2$. In the p-adic metric, the map $\bar{L} = \mathcal{E}L\mathcal{E}^{-1}$ becomes a linear hyperbolic system, driven by a perturbation ι of size $|\iota(x,y)|_p \leq 1$ —cf. (4). Consider now the limit

$$\lim_{n\to\infty} \frac{a + a'p^n}{b + b'p^n} = \begin{cases} a'/b' & \text{in the euclidean metric} \\ a/b & \text{in the } p\text{-adic metric.} \end{cases}$$

This suffices to establish that the image of a rational interval under $\mathcal{E}$ is *dense* in $\mathbb{Q}_p$, and therefore the image of each atom Ω_i is dense in $\mathbb{Q}_p^2$. Thus, even though each rational point z does have a well-defined symbolic dynamics, the function $z \mapsto \iota(z)$, as a p-adic function, is everywhere discontinuous on $\mathbb{Q}^2$. In view of this discontinuity, extending the map $\bar{L}$ from $\mathcal{E}(\Omega_{\mathbb{Q}})$ to $\mathbb{Q}_p^2$ seems quite artificial. More meaningful would be the introduction of a stochastic element, whereby ι is modelled by a stochastic process equipped with

[11] For high-precision calculations, one would instead use the p-adic Newton's method (Hensel's lemma), which is superconvergent.

probabilities given by the measure of the atoms. The difficulty in pursuing this line of investigation derives from our limited knowledge of the symbolic dynamics, which is very highly pruned.

The present situation is quite different from that of the lattice map (3) mentioned in the introduction, and also of the 2-adic representations of the '$3n+1$' dynamical system, where one shows that the corresponding symbolic dynamics function is continuous (albeit nowhere differentiable), leading to a rich p-adic dynamics [7, section 5], [38, chapter 1.10].

6 Experiments

From the data displayed in figures 2, 4 for $\lambda = 1/2$, and from extensive numerical evidence collected at several other parameter values, we put forward the following

Conjecture 1 *For all irrational rotation numbers, the following sets are infinite:*

(i) the set of symmetric stable periodic orbits

(ii) the set of symmetric pseudo-hyperbolic sequences

(iii) the set of asymmetric pseudo-hyperbolic sequences.

By contrast, we found no evidence to suggest that the set of non-symmetric cycles —stable or unstable— and the set of symmetric unstable cycles are infinite. We have seen that unstable cycles correspond to bifurcations, which take place at algebraic parameter values. An infinitude of unstable cycles would then correspond to infinitely many bifurcations occurring simultaneously at one parameter value. This in turn would require a single irreducible polynomial to divide infinitely many elements of the sequence of $\mathcal{X}_t$-polynomials that correspond to valid codes for that parameter. Although this circumstance seems unlikely, we do not have an argument to exclude it. The observed scarcity of stable asymmetric cycles is even more problematic; as for pseudo-hyperbolic sequences, these orbits were expected to dominate the statistics over the symmetric ones, yet the only ones we have observed originated from symmetry-breaking, see below. At the same time, we are not aware of any theoretical obstruction to their infinitude.

In the rest of this section, we articulate some aspects of this phenomenon in relation to a conjecture by Ashwin [3], who considered the so-called *saw-tooth map*

$$F : \Omega \to \Omega \qquad (x, y) \mapsto (x', \{y + x'\}), \qquad x' = \{x + \kappa y - \kappa/2\} \tag{55}$$

which is related to our map L —see equations (56) and (58 below. For parameter values corresponding to irrational rotation numbers, Ashwin presented convincing evidence that the closure of the discontinuity set has non-zero Lebesgue measure. Ashwin's strategy was to partition Ω into boxes of width $\epsilon = 2^{-n}$ and count the number of boxes visited by orbits initiated at and returning to an ϵ-neighbourhood of the discontinuity line $y = 0$. Assuming that the total area covered by the boxes behaves, for $\epsilon \to 0$ as $A - B\epsilon^C$, he found best fits for a multitude of parameter values in the range $-1 \leq \kappa \leq 0$. The limiting total area A was found consistently to fall short of unity for irrational rotation numbers. For $\kappa = -\frac{1}{2}$ the best fit was found to give $A = 0.879$.

An alternative approach consists in identifying the aforementioned measure with the complement of the measure of the cells surrounding the stable cycles. The symmetric cycles are a lot easier to compute than the asymmetric ones, and the absence of the latter becomes a computational blessing. Before pursuing this line of investigation, we relate the saw-tooth map (55) to our family of maps.

6.1 A larger family of maps

We generalize the map L by shifting the fundamental domain Ω along the main diagonal: $\Omega' = \Omega + (\delta, \delta)$. It is not difficult to see that the resulting toral map of Ω' is conjugate to the map

$$L' : \Omega \to \Omega \qquad (x, y) \mapsto (\{\lambda x - y + \eta\}, x) \qquad \eta = \delta(\lambda - 2) \tag{56}$$

and that the map L' is still G-reversible for all values of η. Indeed $L = H'G$ where H' is an orientation-reversing involution given by $H' = L'G : (x, y) \mapsto (\lambda y - x + \eta, y)$ —cf. equation (10).

Consider now the following orientation-reversing involution

$$J : (x, y) \mapsto (1 - y, 1 - x). \tag{57}$$

Letting $E = L'J$, we find

$$E : (x, y) \mapsto (\{-\lambda y + x + \lambda + \eta\}, 1 - y) \qquad E^2 : (x, y) \mapsto (x + 2\eta + \lambda, y).$$

Hence, at the specific parameter value $\eta = -\lambda/2$, E is also an orientation-reversing involution, and therefore the map L' becomes J-reversible on $\Omega\backslash\Gamma$. The lack of J-reversibility on Γ depends on the fact that J has the advertised functional form (57) only on the interior of Ω. This restriction does not concern us, since we are only interested in stable orbits. We find that $\mathrm{Fix}(J)$ is the line $y = 1 - x$, while $\mathrm{Fix}(E)$ is the line $y = 1/2$. Analogous to (12) are the relations

$$z_t \in \mathrm{Fix}(J) \iff x_t = 1 - x_{t-1} \qquad z_t \in \mathrm{Fix}(E) \iff x_t = 1 - x_{t-2}.$$

The map L' with $\eta = -\lambda/2$ corresponds to the non-trivial fixed point being shifted to $(1/2, 1/2)$.

The algorithm for finding periodic orbits described in section 2 applies to G as well as to J. In addition, the symmetric periodic orbits of L' can be classified into orbits symmetric only under G, only under J, and under both. Of interest is the fact that, for fixed λ, any stable orbit which is J-symmetric but not G-symmetric, will become asymmetric for all values of the shift η lying in a sufficiently small neighbourhood of the doubly-symmetric parameter value $\eta = -\lambda/2$. Now, the representation (26) of the periodic points applies to the map L' as well (see remark following the proof of theorem 2), because the η-dependence is buried in the code ι. Therefore, under the assumption that led to the estimate (38) —existence of at least one point in each cycle having maximal degree— the size of the λ-intervals goes to zero as the period increases. Thus, even though asymmetric periodic cycles can be constructed perburbatively near $\eta = -\lambda/2$, by shifting the parameter within the same λ-interval, only *finitely many* of them can survive a perturbation. These are the only asymmetric periodic orbits we have actually observed.

6.2 Numerical test of Ashwin's conjecture

The sawtooth map F of equation (55) is conjugate almost everywhere to the map L' with $\eta = -\lambda/2$, via the conjugacy

$$L' = U^{-1} F U \qquad\qquad U : (x, y) \mapsto (\{x + (\kappa+1)y - \kappa/2\},\, y) \tag{58}$$

together with the reparametrization $\lambda = \kappa + 2$. In particular, the value $\kappa = -1/2$ for the saw-tooth map corresponds to $\lambda = 3/2$ for the map L'. We now test Ashwin's conjecture at this parameter value, by constructing explicitly all stable periodic orbits of L', whose cells have linear size exceeding some small ϵ. Then we fit the total area to a power law $A - B\epsilon^C$. In view of the exponential estimate of proposition 5, computing exact rational points becomes unfeasible for large periods, and we shall resort to floating-point computations, with suitable error control.

In the course of our calculations, we will need to compare the sizes of differently oriented line segments. The invariant length of a segment whose endpoints differ by a vector displacement $(\Delta x, \Delta y)$ is given by the metric

$$(\Delta s)^2 = (\Delta x)^2 - \lambda \Delta x \Delta y + (\Delta y)^2,$$

which is the quadratic form invariant under the matrix S. For an elliptical cell, every point on the boundary is an invariant distance q from the centre, and we refer to $2q$ as its diameter. The cell's area (see (8)) is then $\mathcal{A} = \pi q^2/\sqrt{1 - \lambda^2/4}$.

We begin with symmetric cycle, for which we use the algorithm described at the end of section 2. To find all G-symmetric orbits with a point on the symmetry line $\mathrm{Fix}(G)$, we start with the (open) segment $\mathrm{Fix}(G)$ and apply the map L' iteratively, splitting the segment every time it intersects a discontinuity line. If, at iteration n, one of the image segments intersects transversally $\mathrm{Fix}(G)$, the intersection point z is a stable G-symmetric periodic point of period $2n$; if it intersects $\mathrm{Fix}(H')$, the period is $2n - 1$. With the same procedure we construct the orbits of even period with two points on $\mathrm{Fix}(H')$. The same algorithm is repeated for J-symmetric orbits, using the respective symmetry lines. By mapping around the orbit of each intersection point z, we then determine the area of the cell surrounding z using (8). The intersection with the closure of the cell is then deleted from the segment, leaving two open segments to be further iterated.

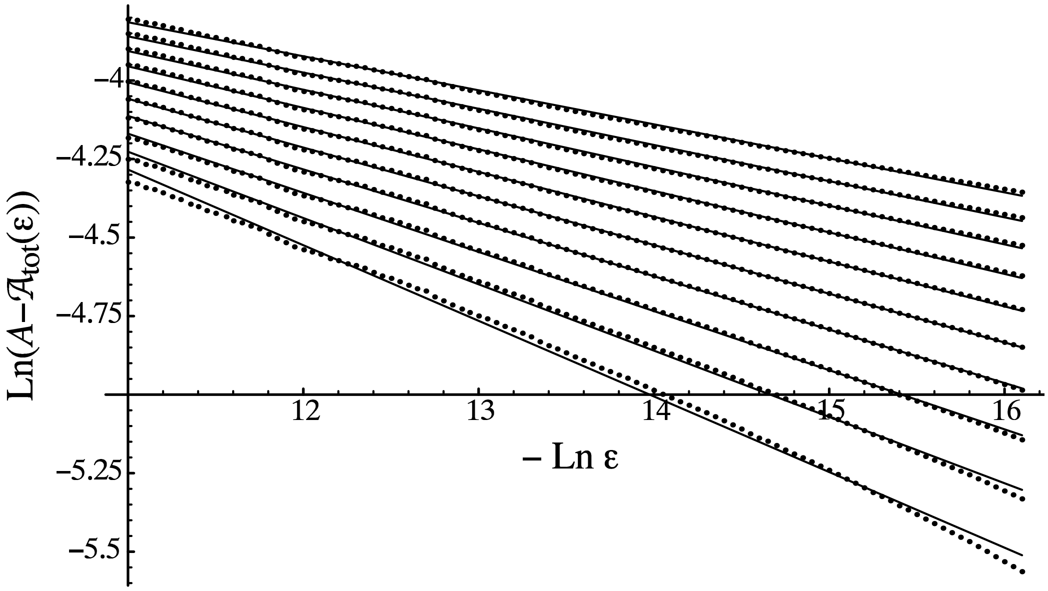

Figure 5: Plot of $\ln(A - \mathcal{A}(\epsilon))$ versus $-\ln\epsilon$ for the map L' with $\lambda = 3/2$ and $\eta = -\lambda/2$, which is conjugate to the saw-tooth map with $\kappa = -1/2$. The same 103 data points, equally spaced in $\ln\epsilon$, are plotted for 10 choices of the area limit A ranging from 0.900 (bottom) to 0.909 (top). In each case the data are least-squares fit to a straight line.

Applying this technique, we are able to harvest systematically large numbers of periodic points and their respective periods and cell areas. Since we are only interested in cells of diameter exceeding ϵ, we can discard segments at any stage whose invariant length is less than ϵ. This pruning is essential for keeping the population of segments to a manageable size.

In our numerical experiment, we iterated the map 10^8 times and collected all G and J-symmetric orbits for $\lambda = \frac{3}{2}$ with cell diameter greater than $\epsilon = 10^{-7}$ (37886 orbits with a maximum period of 28,452,581 and total cell area of 0.896118). In addition, we made a non-exhaustive search for non-symmetric orbits with cells of comparable size. We considered the line $y = \epsilon/2$ (very close to the lower boundary of Ω), deleting all intersections with the cells of symmetric orbits. We iterated the map on the largest gap segments sufficiently many times that any cell of diameter greater than ϵ would have shown up as an unbranching segment. *No stable asymmetric cycles were found.*

The test of Ashwin's conjecture is displayed in figure 5, where we have plotted the natural logarithm of $A - \mathcal{A}_{\rm tot}(\epsilon)$ versus $-\ln\epsilon$ for 10 trial values of A. For each of the latter, 103 approximately equally spaced data points (shown as dots) were fit by a linear function (displayed as a solid line). The best least-squares fit was obtained for $A = 0.904$, with the corresponding power law:

$$\mathcal{A}_{\text{tot}}(\epsilon) = 0.904 - 0.0952\epsilon^{0.155}.$$

A similar experiment was performed for the original map L with its single time-reversal symmetry, for parameter $\lambda = \frac{1}{2}$ and cut-off $\epsilon = 10^{-6}$. The systematic collection of symmetric orbits (9346 orbits, total area 0.792365) was followed by a search for non-symmetric ones, again with a null result. The least-squares fitting led to the power law

$$\mathcal{A}_{\text{tot}}(\epsilon) = 0.817 - 0.508\epsilon^{0.219}.$$

In closing, our data provide independent support to Ashwin's conjecture by showing, at two specific parameter values, results consistent with his, together with evidence of stable asymptotic behaviour. However, piecewise isometries are notoriously prone to slow (logarithmic) relaxation to asymptotics, and one cannot exclude the possibility that our data have not revealed the true asymptotic regime. In particular, the unexplained absence of asymmetric cycles should induce some caution. One approach to this question could involve the study of invariant curves: we found evidence of their existence in certain regions of phase space where the images of the discontinuity lines accumulate at minimal rate —see bottom left corner of figure 1— and where the dynamics seems to be organized around symmetric cycles. We note that a study of invariant curves in a piecewise isometric system was presented recently in [4].

Appendix: some proofs

Proof of identity (15)

We use strong induction on k. The identities

$$S_{t-1}S_t - S_tS_{t-1} = 0 \qquad S_{t-1}S_{t-1} - S_tS_{t-2} = 1, \qquad t \in \mathbb{Z}$$

establish (15) for $k = 0$ and $k = 1$, respectively. The rightmost identity results from the fact that the left-hand side is the determinant of the symplectic matrix

$$S^t = \begin{pmatrix} S_t & -S_{t-1} \\ S_{t-1} & -S_{t-2} \end{pmatrix} \qquad t \in \mathbb{Z} \tag{59}$$

where $S = S^0$ is given in equation (1).

Assume now (15) to be valid for all t and all non-negative k' up to and including some $k \geq 1$. Then, using lemma 1 and the induction hypothesis, we find

$$\begin{aligned}
&S_{t-1}S_{t-(k+1)} - S_tS_{t-1-(k+1)} \\
&\qquad = \lambda S_{t-2}S_{t-k-1} - S_{t-3}S_{t-k-1} - \lambda S_{t-1}S_{t-k-2} + S_{t-2}S_{t-k-2} \\
&\qquad = \lambda(S_{(t-1)-1}S_{(t-1)-k} - S_{(t-1)}S_{(t-1)-1-k}) \\
&\qquad\qquad -(S_{(t-2)-1}S_{(t-2)-(k-1)} - S_{(t-2)}S_{(t-2)-1-(k-1)}) \\
&\qquad = \lambda S_{k-1} - S_{k-2} = S_k
\end{aligned}$$

which completes the induction for positive k. The induction for negative k uses a similar manipulation; we omit the details. □

Proof of identity (25)

The identity holds for $t = 0$, from equations (13) and (20). For $t > 0$, let λ^* be a root of $\mathcal{M}'_{2t+2}$. The matrix $S(\lambda^*)$ given in equation (1) is conjugate to a rotation by the angle $2\pi l/(2t+2)$, for some $l \not\equiv 0, t+1 \pmod{2t+2}$; hence $S^{2t+2}(\lambda^*)$ is the identity, and equation (59) in the appendix gives $S_{2t+2}(\lambda^*) = 1$, $S_{2t+1}(\lambda^*) = 0$. This means that $S_{2t-k}(\lambda^*) = -S_k(\lambda^*)$, which for $k = t$ gives $S_t(\lambda^*) = 0$. Now, the polynomials S_t and $\mathcal{M}'_{2t+2}$ share t roots, have the same degree t, and are monic, so they coincide. □

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