Essential Mathematics

WEB-BOOK *by franco vivaldi*

Preface

Learning modern higher mathematics —discrete mathematics, in particular— requires more than ever fluency in elementary arithmetic and algebra. An alarming number of students reach university without these skills.

The Essential Mathematics programme at Queen Mary is designed to address this problem. It provides training for a compulsory examination, which all first year mathematics students must pass to be admitted to the second year; the exam is to be attempted repeatedly, until passed, and is supported by the course MTH3100. A similar —if a bit less demanding— programme has been introduced as the core course SEF026 within the Science and Engineering foundational year.

The Essential Mathematics programme consists of over 1000 gradual exercises on integers, fractions, radicals, polynomials, rational functions, linear and quadratic equations. Each exercise is provided with answer, for immediate feedback. A student following this programme conscientiously, will develop automatic response, stamina, and confidence, in a vital area of mathematical knowledge.

It would be difficult to overestimate the benefits of this effort.

Are my neurons well-connected?

Look at the following expression for a few seconds, then cover it up.

$$13^2\left(\frac{6}{26}-\frac{8}{39}\right)^2. \tag{1}$$

The goal is to evaluate it *without calculator*. If you can spot and carry out simplifications, this is easy: you do not even need pencil and paper.

So, did you notice any simplification? If you did not, you will probably start off like this

$$13^2\left(\frac{6\cdot 39-26\cdot 8}{26\cdot 39}\right)^2 = 169\left(\frac{234-208}{1014}\right)^2 = \cdots$$

a dangerous move, in an exam without calculator. Consider instead the following equalities

$$26=2\cdot 13; \qquad 39=3\cdot 13; \qquad \left(\frac{a}{d}-\frac{b}{d}\right)^2=\frac{1}{d^2}(a-b)^2 \tag{2}$$

which should be obvious. However, is *their relevance* to the present context equally obvious? Look:

$$13^2 \cdot \left(\frac{6}{2\cdot 13} - \frac{8}{3\cdot 13}\right)^2 = 13^2\,\frac{1}{13^2}\,\left(3-\frac{8}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Simple. But only if, after seeing equation (1), your brain automatically 'connects it' with equations (2), in a process of pattern recognition that barely involves consciousness.

If this has not happened, it means that your mathematical neurons are not well-connected, and you must build these connections. Developing this kind of automatic responses requires little knowledge or understanding, but a lot of regular practice, much like long-distance running, or playing a musical instrument. The only thing to be understood here is that *these skills cannot be acquired in any other way.*

- Work without calculator (which is not permitted in the exam).
- Attempt 5 to 10 exercises a day, every day.
- Do not skip days.
- Do not skip exercises, unless you consistently get the correct answer at the first attempt.
- Discuss any problem with your advisor or lecturer.

Exercises displaying the symbol ** are exam-type questions for MTH3100, while the symbol * flags easier questions, of the type appearing in the exam of the foundational course SEF026. The symbol ♣ denotes a challenge question, which goes beyond what an average student following this programme is expected to handle.

Proof-reading this document has been a labourious process. The original version of the web-book was checked by a team of postgraduate students at Queen Mary, who corrected many errors; I thank them for their effort. I am also grateful to the many undergraduate students that subsequently spotted mistakes while taking the course. Special thanks go to Marcus Jordan, a very young student from Korea, who followed the current version of this programme from beginning to end, resulting in a final set of corrections. The responsibility for any remaining error rests solely with the author.

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Part I: essential arithmetic

I.1 Divisibility and primes

We consider the integers, and the operation of multiplication. We write ab or $a \cdot b$ for $a \times b$. We recall some basic identities concerning multiplication and exponentiation of numbers

$$a\,b = b\,a \qquad (a\,b)^c = a^c\,b^c \qquad a^b a^c = a^{b+c} \qquad \left(a^b\right)^c = a^{bc}.$$

(What is the value of 1^0? Do the expressions 0^1, and 0^0 make sense? Think about it; look it up.)

— — —

Let a and d be integers. We say that d *divides* a if we can write $a = q \cdot d$ for some integer q. The integer $q = a \div d$ is called the *quotient* of the division of a by d. Thus 7 divides 35 because $35 = 5 \cdot 7$, with quotient $5 = 35 \div 7$.

If $d \neq 0$, we can always find integers q and r, such that $a = q \cdot d + r$, with $0 \leq r < |d|$. (The symbol $|\cdot|$ denotes the *absolute value*. What is it?) The integer r is called the *remainder* of the division of a by d. One sees that d divides a precisely when $a \div d$ gives remainder zero. Thus 7 does not divides 37 because $37 \div 7$ gives remainder 2: $37 = 5 \cdot 7 + 2$.

Be careful when a and/or d are negative, because r cannot be negative. Thus dividing -37 by 7 gives $q = -6$ and $r = 5$, because $-37 = -6 \cdot 7 + 5$.

Quotient q and remainder r of division of a by b are computed with the *long division algorithm.* To check your calculation, verify that multiplying q by b and adding r, one recovers a. For example, if $a = 199$ and $b = 13$, one finds $q = 15$ and $r = 4$. Check: $15 \cdot 13 + 4 = 195 + 4 = 199$.

Every integer is divisible by itself and by 1. Every integer divides 0. Some simple criteria exist, to test divisibility by certain integers

d	*d divides a if the digits of a are such that*	a	*d divides a because*
2	2 divides last digit	2008	2 divides 8
3	3 divides sum of digits	10713	3 divides $1+0+7+1+3=12$
4	4 divides last two digits	94152	4 divides 52
5	last digit is 0 or 5	7015	last digit is 5
9	9 divides sum of digits	42381	9 divides $4+2+3+8+1=18$
10	last digit is 0	12340	last digit is 0
11	11 divides sum of digits with alternating signs	752301	11 divides $7-5+2-3+0-1=0$
10^k	last k digits are 0	6070000	last 4 digits are 0 (here $k=4$)

A *prime* is an integer greater than 1 which is divisible only by itself and 1. (So 1 is not prime. Why?) The sequence of primes

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$$

is infinite. There are 15 primes less than 50, and 25 primes less than 100. Every integer greater than 1 can be decomposed as a product of primes, and this decomposition is unique, apart from rearrangement of factors.

$$120 \;=\; 2^3 \cdot 3 \cdot 5 \;=\; 5 \cdot 3 \cdot 2^3 \;=\; 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \;=\; \text{etc.}$$

Writing all exponents explicitly: $120 = 2^3 \cdot 3^1 \cdot 5^1$.

In computations, take advantage of partial factorizations, e.g.,

$$18^3 \cdot 51^5 = (2 \cdot 3^2)^3 \cdot (3 \cdot 17)^5 = 2^3 \cdot 3^6 \cdot 3^5 \cdot 17^5 = 2^3 \cdot 3^{11} \cdot 17^5.$$

To test whether or not a positive integer n is prime, we check divisibility by all primes p with $p \leq \sqrt{n}$. If n is not divisible by any of them, then n is prime. Let $n = 127$. We find $11 < \sqrt{127} < 12$ (see section I.5), and so we must test divisibility by $p = 2, 3, 5, 7, 11$. One verifies that none of these primes divides 127, so 127 is prime.

To find all primes in a given interval $[a, b]$, use the *sieve algorithm.* First list all integers in that interval, excluding those obviously non-prime (i.e., divisible by 2 or 5). So if $a = 141, b = 161$, we list

$$141, 143, 147, 149, 151, 153, 157, 159, 161.$$

Then we eliminate from the list the integers divisible by $3, 7, 11, \ldots, p$, where p is the largest prime such that $p^2 \leq b$ ($p = 11$, in this case). Divisibility by 3 gives:

$$\underline{141}, 143, \underline{147}, 149, 151, \underline{153}, 157, \underline{159}, 161.$$

Next, we eliminate the numbers divisible by 7:

$$\underline{141}, 143, \underline{147}, 149, 151, \underline{153}, 157, \underline{159}, \underline{161}$$

and by 11:

$$\underline{141}, \underline{143}, \underline{147}, 149, 151, \underline{153}, 157, \underline{159}, \underline{161}.$$

There are 3 primes in our interval: $149, 151, 157$.

The above methods for identifying and constructing prime numbers work only with small integers. For intance, the 39-digit integer

$$2^{127} - 1 = 170141183460469231731687303715884105727$$

was proved to be prime in 1878, using a very sophisticated algorithm. This is the largest prime known before the computer age, yet its primality could not be established by our method, even if implemented on the world's fastest computer. (Why? Think about it.)

Compute the quotient and remainder (q, r) of each integer division. Check your calculation explicitly.

1. $1 \div 2$; $3 \div 2$; $12 \div 2$ — $\left[(0,1);\quad (1,1);\quad (6,0)\right]$

2. $7 \div 1$; $2 \div 3$; $2 \div 4$ — $\left[(7,0);\quad (0,2);\quad (0,2)\right]$

3. $0 \div 8$; $8 \div 8$; $24 \div 8$ — $\left[(0,0);\quad (1,0);\quad (3,0)\right]$

4. $20 \div 21$; $25 \div 7$; $32 \div 11$ — $\left[(0,20);\quad (3,4);\quad (2,10)\right]$

5. $40 \div 41$; $57 \div 7$; $80 \div 9$ — $\left[(0,40);\quad (8,1);\quad (8,8)\right]$

6. $200 \div 24$; $91 \div 13$; $175 \div 9$ — $\left[(8,8);\quad (7,0);\quad (19,4)\right]$

7. $111 \div 13$; $197 \div 3$; $222 \div 5$ — $\left[(8,7);\quad (65,2);\quad (44,2)\right]$

8. $1000 \div 1001$; $1001 \div 1000$; $1001 \div 100$ — $\left[(0,1000);\quad (1,1);\quad (10,1)\right]$

9. $177 \div 3$; $1381 \div 2$; $1000 \div 7$ — $\left[(59,0);\quad (690,1);\quad (142,6)\right]$

10.* $500 \div 17$; $499 \div 11$; $309 \div 14$ — $\left[(29,7);\quad (45,4);\quad (22,1)\right]$

11.* $874 \div 12$; $601 \div 21$; $444 \div 11$ — $\left[(72,10);\quad (28,13);\quad (40,4)\right]$

12.* $8195 \div 4$; $3316 \div 9$; $70551 \div 3$ — $\left[(2048,3);\quad (368,4);\quad (23517,0)\right]$

13. $1101 \div 15$; $2001 \div 14$; $729 \div 15$ — $\left[(73,6);\quad (142,13);\quad (48,9)\right]$

14.** $2323 \div 11$; $2121 \div 11$; $9999 \div 12$ — $\left[(211,2);\quad (192,9);\quad (833,3)\right]$

15.** $3421 \div 21$; $10^4 \div 7$; $2091 \div 19$ — $\left[(162,19);\quad (1428,4);\quad (110,1)\right]$

Determine the number of primes lying in the given interval, endpoints included.

16. [0, 2]; [1, 10]; [15, 25] $\left[1;\quad 4;\quad 3\right]$

17. [24, 28]; [30, 40]; [40, 50] $\left[0;\quad 2;\quad 3\right]$

18. [50, 55]; [57, 65]; [60, 90] $\left[1;\quad 2;\quad 7\right]$

19.** [75, 95]; [81, 101]; [39, 69] $\left[3;\quad 4;\quad 7\right]$

20.** [57, 91]; [101, 111]; [111, 125] $\left[8;\quad 4;\quad 1\right]$

Express each integer as a product of prime numbers. Occasionally, check your calculation explicitly.

21. 8; 14; 12 $\left[2^3;\quad 2\cdot 7;\quad 2^2\cdot 3\right]$

22. 1; 11; 21 $\left[\text{impossible};\quad 11;\quad 3\cdot 7\right]$

23. 18; 24; 30 $\left[2\cdot 3^2;\quad 2^3\cdot 3;\quad 2\cdot 3\cdot 5\right]$

24. 27; 49; 81 $\left[3^3;\quad 7^2;\quad 3^4\right]$

25. 25; 50; 100 $\left[5^2;\quad 2\cdot 5^2;\quad 2^2\cdot 5^2\right]$

26. 16; 32; 64 $\left[2^4;\quad 2^5;\quad 2^6\right]$

27. 37; 40; 41 $\left[37;\quad 2^3\cdot 5;\quad 41\right]$

28. 39; 51; 68 $\left[3\cdot 13;\quad 3\cdot 17;\quad 2^2\cdot 17\right]$

29. 57; 58; 59 $\left[3\cdot 19;\quad 2\cdot 29;\quad 59\right]$

30. 65; 66; 72 $\left[5\cdot 13;\quad 2\cdot 3\cdot 11;\quad 2^3\cdot 3^2\right]$

31.* 74; 76; 78 $\left[2\cdot 37;\quad 2^2\cdot 19;\quad 2\cdot 3\cdot 13\right]$

32.* 84; 86; 87 $\left[2^2\cdot 3\cdot 7;\quad 2\cdot 43;\quad 3\cdot 29\right]$

33.* 90; 96; 98 $\left[2\cdot 3^2\cdot 5;\quad 2^5\cdot 3;\quad 2\cdot 7^2\right]$

34.* 121; 128; 150 $\left[11^2;\quad 2^7;\quad 2\cdot 3\cdot 5^2\right]$

35.* 111; 189; 400 $\left[3\cdot 37;\quad 3^3\cdot 7;\quad 2^4\cdot 5^2\right]$

36. $8\cdot 5$; $8\cdot 10$ $\left[2^3\cdot 5;\quad 2^4\cdot 5\right]$

37. $12\cdot 18$; $4\cdot 8\cdot 16$ $\left[2^3\cdot 3^3;\quad 2^9\right]$

38. $6\cdot 30$; $21\cdot 49$ $\left[2^2\cdot 3^2\cdot 5;\quad 3\cdot 7^3\right]$

39. $(27 \cdot 17)^2$; 121^3 $\left[3^6 \cdot 17^2; \quad 11^6\right]$

40. 700000; $(400)^{20}$ $\left[2^5 \cdot 5^5 \cdot 7; \quad 2^{80} \cdot 5^{40}\right]$

41.** 91; 97; 119 $\left[7 \cdot 13; \quad 97; \quad 7 \cdot 17\right]$

42.** 204; 225; 625 $\left[2^2 \cdot 3 \cdot 17; \quad 3^2 \cdot 5^2; \quad 5^4\right]$

43.** 333; 343; 351 $\left[3^2 \cdot 37; \quad 7^3; \quad 3^3 \cdot 13\right]$

44.** 1331; 12^7; $59 \cdot 61$ $\left[11^3; \quad 2^{14} \cdot 3^7; \quad 59 \cdot 61\right]$

45.** 30^4; 36^3; $28 \cdot 45$ $\left[2^4 \cdot 3^4 \cdot 5^4; \quad 2^6 \cdot 3^6; \quad 2^2 \cdot 3^2 \cdot 5 \cdot 7\right]$

46.** 1260; 2220; 1984 $\left[2^2 \cdot 3^2 \cdot 5 \cdot 7; \quad 2^2 \cdot 3 \cdot 5 \cdot 37; \quad 2^6 \cdot 31\right]$

47.** 1863; 2187; 3003 $\left[3^4 \cdot 23; \quad 3^7; \quad 3 \cdot 7 \cdot 11 \cdot 13\right]$

48.** 1000^{1000}; 90^{90} $\left[2^{3000} \cdot 5^{3000}; \quad 2^{90} \cdot 3^{180} \cdot 5^{90}\right]$

♣ 16777216; 16266151 $\left[2^{24}; \quad 11^5 \cdot 101\right]$

I.2 Gcd and lcm

The *greatest common divisor* (gcd — also called *highest common factor*) of two integers is the largest integer dividing both. For example, $\gcd(4,6) = 2$, $\gcd(0,5) = 5$. The value of $\gcd(0,0)$ is undefined (why?), so we let $\gcd(0,0) = 0$. The *least common multiple* (lcm) of two integers is the smallest non-negative integer divisible by both. We illustrate the computation of $\gcd(a,b)$ and $\mathrm{lcm}(a,b)$, with an example: $a = 270$, $b = 252$. First, factor the given integers

$$270 = 2 \cdot 3^3 \cdot 5 \qquad\qquad 252 = 2^2 \cdot 3^2 \cdot 7.$$

Then, match the prime divisors of the two integers by inserting, if necessary, any missing prime raised to the power zero

$$270 = 2^1 \cdot 3^3 \cdot 5^1 \cdot 7^0 \qquad\qquad 252 = 2^2 \cdot 3^2 \cdot 5^0 \cdot 7^1.$$

Now, the gcd is the product of all above primes, each raised to the *smallest* of the two exponents. For the lcm, one instead selects the *largest* of the two exponents.

$$\gcd(270,252) = 2^1 \cdot 3^2 \cdot 5^0 \cdot 7^0 = 2 \cdot 9 = 18; \qquad \mathrm{lcm}(270,252) = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^1 = 4 \cdot 27 \cdot 35 = 3780.$$

Persuade yourself that if a divides b, then $\gcd(a,b) = a$ and $\mathrm{lcm}(a,b) = b$. For instance

$$\gcd(21,63) = 21 \qquad\qquad \mathrm{lcm}(21,63) = 63.$$

Two integers are *relatively prime* if their greatest common divisor is 1. The least common multiple of two relatively prime integers is equal to their product. (Think about it.) Do not confuse the terms *prime* and *relatively prime.* Two distinct primes are necessarily relatively prime, but two integers can be relatively prime without any of them being prime. For instance

$$1 = \gcd(4,9) = \gcd(14,15) = \gcd(27,10) = \gcd(16,33).$$

Compute the greatest common divisor of the following pairs of integers.

1. $(1,2)$; $(-2,2)$; $(2,4)$ $\qquad [1;\ 2;\ 2]$

2. $(0,3)$; $(2^2,2^3)$; $(-6,-8)$ $\qquad [3;\ 4;\ 2]$

3. $(21,18)$; $(20,18)$; $(19,18)$ $\qquad [3;\ 2;\ 1]$

4.* $(33,51)$; $(34,51)$; $(35,51)$ $\qquad [3;\ 17;\ 1]$

5.* $(60,84)$; $(72,90)$; $(45,63)$ $\qquad [12;\ 18;\ 9]$

6.* $(30,78)$; $(55,20)$; $(60,75)$ $\qquad [6;\ 5;\ 15]$

7. $(70,84)$; $(-77,60)$; $(96,128)$ $\qquad [14;\ 1;\ 32]$

8. $(108, 180)$; $(270, 252)$; $(140, 200)$ $\big[36;\ 18;\ 20\big]$

9.** $(315, 675)$; $(630, 588)$; $(900, 240)$ $\big[45;\ 42;\ 60\big]$

10.** $(594, 693)$; $(432, 882)$; $(450, 864)$ $\big[99;\ 18;\ 18\big]$

11.** $(240, 18^2)$; $(21^3, 315)$; $(84^2, 14^5)$ $\big[12;\ 63;\ 784\big]$

12.** $(12^2, 90^4)$; $(48^3, 51)$; $(77^3, 242)$ $\big[144;\ 3;\ 121\big]$

13.** $(9^9, 141^2)$; $(66, 18^5)$; $(26^4, 2600)$ $\big[9;\ 6;\ 104\big]$

Compute the least common multiple of the following pairs of integers.

14. $(2, 10)$; $(2, 5)$; $(6, 8)$ $\big[10;\ 10;\ 24\big]$

15. $(8, 12)$; $(20, 30)$; $(12, 18)$ $\big[24;\ 60;\ 36\big]$

16.* $(11, 11)$; $(11, 13)$; $(12, 15)$ $\big[11;\ 143;\ 60\big]$

17.* $(72, 68)$; $(24, 30)$; $(25, 40)$ $\big[1224;\ 120;\ 200\big]$

18. $(54, 60)$; $(42, 45)$; $(70, 75)$ $\big[540;\ 630;\ 1050\big]$

19.** $(84, 63)$; $(60, 72)$; $(48, 56)$ $\big[252;\ 360;\ 336\big]$

20.** $(180, 240)$; $(198, 242)$; $(135, 315)$ $\big[720;\ 2178;\ 945\big]$

Let G and L be the greatest common divisor and least common multiple, respectively, of the following pairs of integers. Compute $L - G$.

21. $(2, 3)$; $(2, 4)$; $(10, 10)$ $\big[5;\ 2;\ 0\big]$

22. $(2, 10)$; $(14, 7)$; $(49, 7)$ $\big[8;\ 7;\ 42\big]$

23.* $(18, 32)$; $(12, 33)$; $(20, 30)$ $\big[286;\ 129;\ 50\big]$

24.** $(48, 50)$; $(56, 44)$; $(36, 168)$ $\big[1198;\ 612;\ 492\big]$

25.** $(75, 100)$; $(6, 111)$; $(30, 64)$ $\big[275;\ 219;\ 958\big]$

26.** $(60, 84)$; $(36, 24)$; $(65, 91)$ $\big[408;\ 60;\ 442\big]$

27.** $(55, 66)$; $(91, 14)$; $(76, 57)$ $\big[319;\ 175;\ 209\big]$

♣ $(3003, 3230)$ $\big[9699689\big]$

I.3 Fractions

A *fraction* is the ratio of two integers, the numerator and the denominator, where the denominator is non-zero. An integer may be thought of as a fraction with denominator 1. A fraction is *reduced* if numerator and denominator are relatively prime, and the denominator is positive.

$$\frac{19}{54} \text{ is reduced;} \qquad \frac{36}{54} \text{ is not reduced, because } \frac{36}{54} = \frac{2 \cdot 18}{3 \cdot 18} = \frac{2}{3}.$$

To reduce a fraction, one divides numerator and denominator by their *greatest common divisor.* With reference to the above example, $\gcd(36, 54) = 18$.

The following identities are helpful in handling negative signs:

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}. \tag{I.3.1}$$

Equality of fractions is defined in terms of equality of integers, as follows:

$$\frac{a}{b} = \frac{c}{d} \qquad \Longleftrightarrow \qquad a\,d = b\,c \qquad\qquad b, d \neq 0$$

and similarly for $\neq$. Inequality of fractions is handled similarly:

$$\frac{a}{b} < \frac{c}{d} \qquad \Longleftrightarrow \qquad a\,d < b\,c \qquad\qquad bd > 0,$$

and similarly for $\leq$, $>$, $\geq$. For example

$$\frac{144}{233} < \frac{89}{144} \qquad \text{because} \qquad 144 \times 144 = 20736 < 20737 = 233 \times 89.$$

In this example, the closeness of the integers in the right inequality (they differ by one part in twenty thousand) translates into the closeness of the fractions in the left inequality (they differ at the 5th decimal place)

$$\frac{144}{233} = 0.618025751\ldots < 0.618055555\ldots = \frac{89}{144}.$$

Every fraction can be written as the sum of an integer and a fraction lying between 0 (included) and 1 (excluded), which are its *integer* and *fractional part*, respectively. The fractional part of an integer is zero.

$$\frac{19}{5} = 3 + \frac{4}{5} \qquad\qquad \frac{21}{6} = 3 + \frac{1}{2} \qquad\qquad \frac{6}{21} = 0 + \frac{2}{7}.$$

We discourage using the notation $3\frac{4}{5}$ for $3 + \frac{4}{5}$.

The integer part of a fraction is the quotient of division of numerator by denominator; the fractional part is the remainder of the same division, divided by the denominator. Thus integer and fractional part are computed via the *long division* algorithm. Before performing long division, make sure that the fraction is reduced. For instance, in the second example above, the fraction $21/6$ is first reduced to $7/2$; the division $7 \div 2$ then gives quotient 3 and remainder 1, hence the result.

Knowledge of the integer part of a fraction affords a crude estimate of the fraction's size

$$\frac{1744}{23} = 75 + \frac{19}{23} \qquad \Longrightarrow \qquad 75 < \frac{1744}{23} < 76.$$

Because $19/23 > 1/2$, the integer nearest to $1744/23$ is 76.

A number with terminating decimals can be written as a reduced fraction, as follows

$$7.1 = 71 \times 10^{-1} = \frac{71}{10} \qquad\qquad 7.04 = \frac{704}{100} = \frac{176}{25}.$$

However, a reduced fraction has terminating decimals only when 2 and 5 are the only primes which divide the denominator

$$\frac{1}{1280} = \frac{1}{2^8 \times 5} = 0.00078125 \qquad\qquad \frac{1}{7} = 0.142857\,142857\,142857\,142857\,\ldots.$$

Thus most fractions do *not* have terminating decimals.

Simplify.

1. $\frac{6}{4}$; $\frac{16}{4}$; $\frac{9}{12}$ $\qquad\left[\frac{3}{2};\quad 4;\quad \frac{3}{4}\right]$

2. $\frac{6}{-4}$; $\frac{-8}{-12}$; $\frac{13}{-1}$ $\qquad\left[-\frac{3}{2};\quad \frac{2}{3};\quad -13\right]$

3. $\frac{21}{15}$; $\frac{25}{20}$; $\frac{21}{28}$ $\qquad\left[\frac{7}{5};\quad \frac{5}{4};\quad \frac{3}{4}\right]$

4. $\frac{13}{15}$; $\frac{11}{7}$; $\frac{12}{11}$ $\qquad\left[\frac{13}{15};\quad \frac{11}{7};\quad \frac{12}{11}\right]$

5. $\frac{22}{33}$; $\frac{63}{56}$; $\frac{70}{60}$ $\qquad\left[\frac{2}{3};\quad \frac{9}{8};\quad \frac{7}{6}\right]$

6. $\frac{15}{35}$; $\frac{72}{54}$; $\frac{27}{99}$ $\qquad\left[\frac{3}{7};\quad \frac{4}{3};\quad \frac{3}{11}\right]$

7. $\frac{180}{450}$; $\frac{700}{420}$; $\frac{240}{360}$ $\qquad\left[\frac{2}{5};\quad \frac{5}{3};\quad \frac{2}{3}\right]$

8. $\frac{84}{60}$; $\frac{96}{93}$; $\frac{97}{93}$ $\qquad\left[\frac{7}{5};\quad \frac{32}{31};\quad \frac{97}{93}\right]$

9. $\frac{91}{63}$; $\frac{216}{264}$; $\frac{112}{118}$ $\qquad\left[\frac{13}{9};\quad \frac{9}{11};\quad \frac{56}{59}\right]$

Decompose into integer and fractional parts. Simplify first.

10. $\frac{3}{2}$; $\frac{6}{2}$; $\frac{2}{6}$ $\left[1 + \frac{1}{2}; \quad 3 + 0; \quad 0 + \frac{1}{3}\right]$

11. $\frac{1}{13}$; $\frac{14}{13}$; $\frac{27}{13}$ $\left[0 + \frac{1}{13}; \quad 1 + \frac{1}{13}; \quad 2 + \frac{1}{13}\right]$

12. $\frac{20}{18}$; $\frac{33}{2}$; $\frac{27}{7}$ $\left[1 + \frac{1}{9}; \quad 16 + \frac{1}{2}; \quad 3 + \frac{6}{7}\right]$

13.* $\frac{42}{16}$; $\frac{30}{21}$; $\frac{48}{14}$ $\left[2 + \frac{5}{8}; \quad 1 + \frac{3}{7}; \quad 3 + \frac{3}{7}\right]$

14.* $\frac{41}{3}$; $\frac{100}{7}$; $\frac{91}{14}$ $\left[13 + \frac{2}{3}; \quad 14 + \frac{2}{7}; \quad 6 + \frac{1}{2}\right]$

15. $\frac{2001}{1000}$; $\frac{2001}{10000}$ $\left[2 + \frac{1}{1000}; \quad 0 + \frac{2001}{10000}\right]$

16.** $\frac{1848}{39}$; $\frac{1821}{21}$ $\left[47 + \frac{5}{13}; \quad 86 + \frac{5}{7}\right]$

17.** $\frac{1000}{13}$; $\frac{1011}{12}$ $\left[76 + \frac{12}{13}; \quad 84 + \frac{1}{4}\right]$

Determine the integer nearest to the given fraction.

18. $\frac{1}{3}$; $\frac{2}{3}$; $\frac{4}{3}$ $[0; \quad 1; \quad 1]$

19. $\frac{11}{10}$; $\frac{21}{10}$; $\frac{109}{10}$ $[1; \quad 2; \quad 11]$

20. $\frac{13}{5}$; $-\frac{13}{7}$; $\frac{103}{11}$ $[3; \quad -2; \quad 9]$

21.* $\frac{75}{16}$; $-\frac{75}{17}$; $\frac{750}{200}$ $[5; \quad -4; \quad 4]$

22.* $\frac{111}{7}$; $\frac{444}{28}$; $\frac{777}{49}$ $[16; \quad 16; \quad 16]$

23. $\frac{123}{122}$; $\frac{122}{123}$; $\frac{12345}{12346}$ $[1; \quad 1; \quad 1]$

24. $\frac{203}{59}$; $\frac{201}{23}$; $\frac{533}{13}$ $[3; \quad 9; \quad 41]$

25.** $\frac{3528}{24}$; $\frac{1110}{21}$; $\frac{1504}{11}$ $[147; \quad 53; \quad 137]$

26.** $\frac{581}{19}$; $-\frac{4016}{28}$; $\frac{1001}{32}$ $\left[31;\quad -143;\quad 31\right]$

27.** $\frac{1759}{14}$; $\frac{1617}{18}$; $\frac{2888}{29}$ $\left[126;\quad 90;\quad 100\right]$

28.** $\frac{1994}{17}$; $\frac{2001}{23}$; $\frac{11011}{3}$ $\left[117;\quad 87;\quad 3670\right]$

29.** $\frac{50001}{20000}$; $\frac{50000}{20001}$; $\frac{1025}{256}$ $\left[3;\quad 2;\quad 4\right]$

Sort in ascending order.

30. $\frac{2}{3}, \frac{3}{5}$; $\frac{7}{5}, \frac{13}{9}$ $\left[\frac{3}{5} < \frac{2}{3};\quad \frac{7}{5} < \frac{13}{9}\right]$

31. $\frac{21}{13}, \frac{8}{5}$; $\frac{16}{3}, \frac{23}{4}$ $\left[\frac{8}{5} < \frac{21}{13};\quad \frac{16}{3} < \frac{23}{4}\right]$

32. $\frac{200}{7}, 29$; $\frac{41}{3}, \frac{66}{5}$ $\left[\frac{200}{7} < 29;\quad \frac{66}{5} < \frac{41}{3}\right]$

33. $\frac{3}{2}, \frac{7}{5}, \frac{13}{10}$ $\left[\frac{13}{10} < \frac{7}{5} < \frac{3}{2}\right]$

34. $\frac{3}{7}, \frac{5}{9}, \frac{6}{13}$ $\left[\frac{3}{7} < \frac{6}{13} < \frac{5}{9}\right]$

35. $\frac{7}{4}, \frac{12}{7}, \frac{9}{5}$ $\left[\frac{12}{7} < \frac{7}{4} < \frac{9}{5}\right]$

36.* $\frac{5}{8}, \frac{8}{13}, \frac{13}{21}$ $\left[\frac{8}{13} < \frac{13}{21} < \frac{5}{8}\right]$

37.** $\frac{22}{7}, \frac{31}{10}, \frac{19}{6}$ $\left[\frac{31}{10} < \frac{22}{7} < \frac{19}{6}\right]$

38.** $\frac{35}{3}, \frac{58}{5}, \frac{93}{8}$ $\left[\frac{58}{5} < \frac{93}{8} < \frac{35}{3}\right]$

♣ $\frac{4181}{6765}, \frac{6765}{10946}, \frac{10946}{17711}$ $\left[\frac{6765}{10946} < \frac{10946}{17711} < \frac{4181}{6765}\right]$

Write each decimal number as a reduced fraction.

39. 1.5; 0.75; 1.2 $\left[\frac{3}{2};\quad \frac{3}{4};\quad \frac{6}{5}\right]$

40. 0.125; 2.2; 0.01 $\left[\frac{1}{8};\quad \frac{11}{5};\quad \frac{1}{100}\right]$

41. 0.15; 0.02; 1.02 $\left[\dfrac{3}{20};\quad \dfrac{1}{50};\quad \dfrac{51}{50}\right]$

42. 8.1; 8.2; 0.08 $\left[\dfrac{81}{10};\quad \dfrac{41}{5};\quad \dfrac{2}{25}\right]$

43.* 6.04; 0.605; 0.0025 $\left[\dfrac{151}{25};\quad \dfrac{121}{200};\quad \dfrac{1}{400}\right]$

44.* 50.1; 0.051; 0.0015 $\left[\dfrac{501}{10};\quad \dfrac{51}{1000};\quad \dfrac{3}{2000}\right]$

45.* 3.56; 8.96; 0.512 $\left[\dfrac{89}{25};\quad \dfrac{224}{25};\quad \dfrac{64}{125}\right]$

46.** 3.141592; 0.000128 $\left[\dfrac{392699}{125000};\quad \dfrac{2}{15625}\right]$

♣ 0.0009765625 $\left[\dfrac{1}{1024}\right]$

I.4 Arithmetic of fractions

We define arithmetical operations among fractions. Addition and subtraction:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \qquad\qquad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

Multiplication and division:

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd} \qquad\qquad \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

Exponentiation:

$$\left(\frac{a}{b}\right)^{e}=\frac{a^e}{b^e} \qquad\qquad \left(\frac{a}{b}\right)^{-e}=\frac{b^e}{a^e}.$$

When adding/subtracting fractions whose denominators are *not* relatively prime (their gcd is greater than 1 —see section I.2), one should use the reduced formula

$$\frac{a}{b}\pm\frac{c}{d} = \frac{a\,\dfrac{L}{b}\pm c\,\dfrac{L}{d}}{L}; \qquad\qquad L=\operatorname{lcm}(b,d).$$

The following example illustrates the usefulness of the reduced formula

$$\frac{8}{21}-\frac{9}{28}=\frac{8\cdot 4-9\cdot 3}{84}=\frac{5}{84} \qquad\qquad 84=\operatorname{lcm}(21,28).$$

The same calculation, with the basic formula, would give

$$\frac{8}{21}-\frac{9}{28}=\frac{8\cdot 28-9\cdot 21}{588}=\frac{35}{588}=\frac{5}{84} \qquad\qquad 588=21\cdot 28.$$

Before multiplying fractions, check the possibility of cross-simplification

$$\frac{\not{a}}{b}\times\frac{c}{\not{a}}=\frac{c}{b}; \qquad\qquad \frac{40}{51}\times\frac{17}{80} = \frac{1}{6}.$$

Make sure you are fully comfortable with the behaviour of fractions with respect to sign change —cf. equations (I.3.1)

$$-\frac{3-2}{11-7}=-\left(\frac{3-2}{11-7}\right)=\frac{-(3-2)}{11-7}=\frac{3-2}{-(11-7)}=\frac{2-3}{11-7}=\frac{3-2}{7-11}.$$

When evaluating arithmetical expressions involving several operators, multiplication and division are performed before addition and subtraction, e.g.,

$$\frac{a}{b}-\frac{c}{d}\div\frac{e}{f}=\frac{a}{b}-\left(\frac{c}{d}\div\frac{e}{f}\right)=\frac{a}{b}-\frac{cf}{de}=\cdots$$

Expressions within parentheses are evaluated first; with nested parentheses, evaluation begins from the innermost one

$$\left[\frac{1}{4}\times\left(2-\frac{2}{3}\right)\right]\div 5-\left(\frac{2}{5}-\frac{7}{3}\right) = \left[\frac{1}{4}\times\frac{4}{3}\right]\times\frac{1}{5}-\left(-\frac{29}{15}\right) = \frac{1}{15}+\frac{29}{15} = \frac{30}{15} = 2.$$

Keep fractions simplified at every stage of a calculation

Evaluate, eliminating parentheses first.

1. $20 \div 5 - 7 + 3 \times 9$ $\quad [24]$
2. $-1 + 5 + 9 \times 2 - 6 + 30 \div 6$ $\quad [21]$
3. $-7 - 3 \times 5 + 2 - 5$ $\quad [-25]$
4. $35 \div 7 + 2 \times 8 - 3 - 18$ $\quad [0]$
5. $3 \times 9 + 56/8 - 20$ $\quad [14]$
6. $-50 \div 5 - 7 - 2 \times 9 + 7 \times 5$ $\quad [0]$
7. $4 + 42/6 - 35/7 + 10 - 11$ $\quad [5]$
8. $(27 - 7 + 5) - (10 + 3 - 2) + 1$ $\quad [15]$
9. $21 + (11 - 9 + 3) - 10 - (3 + 15 - 11) + 8$ $\quad [17]$
10. $16 + (-3 + 9 + 4) - (7 + 2) - 7$ $\quad [10]$
11. $49 - [21 + (7 - 3 - 1) - 2] + (4 + 7 - 2)$ $\quad [36]$
12. $10 + [7 - (3 - 2 + 4) + (8 - 5)] - (20 + 2 - 18 - 1)$ $\quad [12]$
13. $19 + [(15 - 5 + 2) - 8 + (8 + 4 - 9) - 7] - 19$ $\quad [0]$
14. $[21 + (10 + 3 - 8)] - [10 + (8 + 4 - 5) - 8] - 7$ $\quad [10]$
15. $(18 + 18 \div 3 - 8) + (15 - 27 \div 3 - 2)$ $\quad [20]$
16. $(13 + 16 \div 2 - 8) - (5 + 7 \times 3 - 20)$ $\quad [7]$
17. $15 - (4 + 25/5 - 3) + (7 \times 2 - 10/5)$ $\quad [21]$

Multiply, cross-simplifying first.

18. $\dfrac{3}{4} \times \dfrac{3}{5}; \quad \dfrac{3}{9} \times \dfrac{6}{3}$ $\quad \left[\dfrac{9}{20}; \quad \dfrac{2}{3}\right]$
19. $\dfrac{4}{7} \times \dfrac{8}{28}; \quad \dfrac{6}{8} \times \dfrac{9}{4}$ $\quad \left[\dfrac{8}{49}; \quad \dfrac{27}{16}\right]$
20. $\dfrac{2}{22} \times \dfrac{11}{7}; \quad \dfrac{5}{18} \times \dfrac{6}{11}$ $\quad \left[\dfrac{1}{7}; \quad \dfrac{5}{33}\right]$

21. $\frac{32}{40} \times \frac{50}{16}; \quad \frac{20}{45} \times \frac{15}{10}$ $\left[\frac{5}{2}; \quad \frac{2}{3}\right]$

22. $\frac{3}{101} \times \frac{101}{5}; \quad \frac{65}{29} \times \frac{58}{13}$ $\left[\frac{3}{5}; \quad 10\right]$

23. $\frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8}$ $\left[\frac{1}{4}\right]$

24. $\frac{2}{3} \times \frac{5}{7} \times \frac{11}{13}$ $\left[\frac{110}{273}\right]$

25. $\frac{24}{5} \times \frac{81}{30} \times \frac{25}{36} \times \frac{2}{45}$ $\left[\frac{2}{5}\right]$

Divide.

26. $\frac{6}{11} \div \frac{2}{3}; \quad \frac{2}{5} \div \frac{4}{9}$ $\left[\frac{9}{11}; \quad \frac{9}{10}\right]$

27. $\frac{7}{10} \div \frac{7}{5}; \quad \frac{6}{5} \div \frac{8}{15}$ $\left[\frac{1}{2}; \quad \frac{9}{4}\right]$

28. $\frac{3}{8} \div \frac{9}{7}; \quad \frac{8}{7} \div \frac{4}{3}$ $\left[\frac{7}{24}; \quad \frac{6}{7}\right]$

29. $\frac{2}{3} \div \frac{6}{11}; \quad \frac{9}{10} \div \frac{5}{7}$ $\left[\frac{11}{9}; \quad \frac{63}{50}\right]$

30. $\frac{18}{44} \div \frac{9}{11}; \quad \frac{8}{26} \div \frac{12}{52}$ $\left[\frac{1}{2}; \quad \frac{4}{3}\right]$

Add, subtract.

31. $2 + \frac{1}{3}; \quad \frac{1}{2} + \frac{1}{3}; \quad \frac{1}{8} + \frac{1}{12}$ $\left[\frac{7}{3}; \quad \frac{5}{6}; \quad \frac{5}{24}\right]$

32. $\frac{3}{8} - \frac{1}{4}; \quad \frac{11}{15} + \frac{3}{5}; \quad \frac{3}{4} - \frac{5}{12}$ $\left[\frac{1}{8}; \quad \frac{4}{3}; \quad \frac{1}{3}\right]$

33. $\frac{6}{12} + \frac{10}{20}; \quad \frac{14}{42} + \frac{11}{33}; \quad \frac{13}{39} + \frac{39}{13}$ $\left[1; \quad \frac{2}{3}; \quad \frac{10}{3}\right]$

34. $\frac{17}{119} - \frac{57}{19}; \quad -\frac{96}{84} + \frac{155}{31}$ $\left[-\frac{20}{7}; \quad \frac{27}{7}\right]$

35. $-\frac{1}{4} + \frac{4}{5} - \frac{2}{3} + \frac{5}{6}; \quad \frac{3}{4} + \frac{5}{12} - \frac{1}{20} - \frac{1}{4}$ $\left[\frac{43}{60}; \quad \frac{13}{15}\right]$

36. $\frac{1}{2} + \frac{24}{6} - 2 + \frac{75}{25}; \quad 2 - \frac{1}{4} - \frac{2}{16} - \frac{20}{9}$ $\left[\frac{11}{2}; \quad -\frac{43}{72}\right]$

37. $-7-\dfrac{252}{420}+\dfrac{203}{29}+\dfrac{16}{80}+\dfrac{74}{185}$ $\Big[0\Big]$

Evaluate, eliminating parentheses first.

38. $\dfrac{4}{5}-\left(\dfrac{3}{4}-\dfrac{1}{2}\right)+\dfrac{1}{5}$ $\Big[\dfrac{3}{4}\Big]$

39. $-\dfrac{9}{2}-\left(\dfrac{3}{7}+\dfrac{5}{3}-\dfrac{7}{6}\right)$ $\Big[-\dfrac{38}{7}\Big]$

40. $\left(\dfrac{6}{16}-\dfrac{1}{12}\right)-\left(\dfrac{1}{30}-\dfrac{1}{5}\right)$ $\Big[\dfrac{11}{24}\Big]$

41. $\left(\dfrac{7}{12}-\dfrac{1}{3}-\dfrac{1}{4}\right)-\left(-\dfrac{5}{6}+\dfrac{1}{2}-\dfrac{3}{4}\right)-\left(\dfrac{1}{4}-\dfrac{1}{2}\right)$ $\Big[\dfrac{4}{3}\Big]$

42. $-\dfrac{1}{5}+\left(-\dfrac{3}{2}+\dfrac{1}{4}\right)-\left(\dfrac{3}{4}+\dfrac{1}{5}\right)$ $\Big[-\dfrac{12}{5}\Big]$

43. $-\dfrac{7}{6}+\left(-\dfrac{8}{3}-\dfrac{5}{2}+\dfrac{9}{4}\right)-\left(-\dfrac{9}{2}-\dfrac{1}{4}+\dfrac{2}{3}\right)$ $\Big[0\Big]$

44. $-\dfrac{3}{8}+\left(\dfrac{1}{2}-\dfrac{5}{6}\right)-\left(\dfrac{7}{4}-\dfrac{1}{3}\right)-\dfrac{7}{6}+\dfrac{1}{8}$ $\Big[-\dfrac{19}{6}\Big]$

45. $\dfrac{4}{3}-\left(\dfrac{2}{5}-\dfrac{1}{2}\right)-\dfrac{7}{5}+\left(\dfrac{3}{2}-\dfrac{2}{3}\right)+\dfrac{1}{3}$ $\Big[\dfrac{6}{5}\Big]$

46. $-\dfrac{13}{15}-\left(-\dfrac{3}{4}-\dfrac{1}{2}\right)+\dfrac{3}{5}+\left(-\dfrac{1}{2}+\dfrac{5}{6}-\dfrac{1}{3}\right)$ $\Big[\dfrac{59}{60}\Big]$

47. $-\left[\left(\dfrac{8}{5}+\dfrac{11}{30}-\dfrac{8}{15}\right)-\left(\dfrac{17}{10}-\dfrac{7}{20}\right)\right]+\dfrac{5}{12}$ $\Big[\dfrac{1}{3}\Big]$

Evaluate.

48. $2\times\left(\dfrac{1}{2}+\dfrac{1}{4}\right);\qquad 4\times\left(\dfrac{10}{20}+\dfrac{100}{400}\right)\times\dfrac{1}{2}$ $\Big[\dfrac{3}{2};\quad \dfrac{3}{2}\Big]$

49. $\dfrac{8}{14}\times\left(\dfrac{2}{3}-\dfrac{3}{8}\right);\qquad \left(\dfrac{3}{4}-\dfrac{8}{24}\right)\times\dfrac{24}{5}$ $\Big[\dfrac{1}{6};\quad 2\Big]$

50. $4\times\left(\dfrac{6}{4}-\dfrac{5}{20}\right);\qquad \left(\dfrac{4}{10}+\dfrac{3}{4}\right)\times\dfrac{20}{46}$ $\Big[5;\quad \dfrac{1}{2}\Big]$

51. $\left(\dfrac{2}{5}+\dfrac{3}{5}+\dfrac{2}{15}\right)\times\dfrac{10}{7};\qquad \dfrac{2}{7}\times\left(\dfrac{4}{5}+\dfrac{1}{2}-\dfrac{7}{10}\right)$ $\Big[\dfrac{34}{21};\quad \dfrac{6}{35}\Big]$

52. $\left(\dfrac{3}{10}+\dfrac{6}{5}\right)\times\left(4-\dfrac{7}{28}\right);\qquad \left(2+\dfrac{15}{18}\right)\div\left(6-\dfrac{3}{4}\right)$ $\Big[\dfrac{45}{8};\quad \dfrac{34}{63}\Big]$

53. $\left(\frac{9}{2}-\frac{2}{5}\right)\times\left(3+\frac{12}{36}\right);\qquad \left(\frac{3}{10}+\frac{6}{5}\right)\times\left(4-\frac{17}{68}\right)$ $\left[\frac{41}{3};\quad \frac{45}{8}\right]$

54. $\left(-\frac{40}{50}-\frac{50}{40}-\frac{90}{200}\right)\times\left(\frac{11}{55}-\frac{22}{33}-\frac{11}{165}\right)$ $\left[\frac{4}{3}\right]$

55. $\left(\frac{8}{10}-\frac{15}{9}+\frac{10}{75}\right)\times\left(-1-\frac{28}{77}\right)$ $\left[1\right]$

56. $\left(\frac{3}{4}-\frac{1}{12}-\frac{5}{8}\right)\times\frac{18}{5}+\left(\frac{13}{15}-\frac{1}{6}+1-\frac{1}{10}\right)\times\frac{5}{16}$ $\left[\frac{13}{20}\right]$

57. $\left(\frac{3}{4}-\frac{1}{2}\right)\times 4-\left(\frac{2}{3}-\frac{1}{2}\right)\times\frac{6}{5}-\frac{4}{5}+\left[\frac{12}{15}\times\left(2+\frac{1}{7}\right)-\frac{12}{7}\right]$ $\left[0\right]$

58. $\left[\left(\frac{1}{4}+\frac{3}{2}-\frac{7}{5}+1\right)+\left(\frac{1}{24}+\frac{1}{30}+\frac{1}{20}\right)\times\frac{4}{5}-\frac{2}{5}\right]\times\frac{10}{7}$ $\left[\frac{3}{2}\right]$

Evaluate.

59. $\frac{1}{-1};\qquad \frac{(-1)^2}{-1^2};\qquad -\frac{-1^3}{(-1)^3}$ $\left[-1;\quad -1;\quad -1\right]$

60. $\left(\frac{2}{3}\right)^2;\qquad \left(\frac{5}{7}\right)^0;\qquad \left(\frac{1}{3}\right)^{-2}$ $\left[\frac{4}{9};\quad 1;\quad 9\right]$

61. $\left(2-\frac{1}{3}\right)^3;\qquad \left(\frac{5}{2}-\frac{1}{3}\right)^2;\qquad \left(-\frac{7}{3}+3\right)^4$ $\left[\frac{125}{27};\quad \frac{169}{36};\quad \frac{16}{81}\right]$

62. $\left(-1-\frac{5}{6}\right)^{-2};\qquad \left(\frac{2}{3}-\frac{1}{6}\right)^{-5};\qquad \left(\frac{6}{8}-\frac{6}{10}\right)^{-3}$ $\left[\frac{36}{121};\quad 32;\quad \frac{8000}{27}\right]$

63.* $\left(\frac{2}{3}\right)^{10}\div\left(\frac{2}{3}\right)^8;\qquad \frac{3^5}{2^5}\div\frac{3^6}{2^6};\qquad \left[\left(\frac{2}{5}\right)^4\times\left(\frac{2}{5}\right)^3\right]^2\div\left(\frac{2}{5}\right)^{12}$ $\left[\frac{4}{9};\quad \frac{2}{3};\quad \frac{4}{25}\right]$

Evaluate.

64.* $\frac{4^2}{3^2}\times\frac{3}{20}+\left(\frac{1}{2}+\frac{5}{4}\right)^2\div\frac{7}{16}+\frac{13}{75}-\left(1-\frac{1}{2}\right)^2\times\left(3-\frac{3}{5}\right)^2$ $\left[6\right]$

65.* $\left\{\left(-\frac{1}{9}+\frac{2}{3}\right)^2\div\left[\frac{3}{5}\div\left(2-\frac{1}{5}\right)\right]^4\right\}\times\left(\frac{1}{5}\right)^3+3$ $\left[\frac{16}{5}\right]$

66. $\left[\left(-4+\frac{7}{5}\right)\div\left(2-\frac{7}{5}\right)+\left(-\frac{1}{2}+\frac{7}{3}-3\right)\div\frac{5}{2}\right]\div\left(-\frac{7}{15}+1\right)$ $\left[-9\right]$

67. $\left(-\frac{1}{5}-\frac{7}{25}\right)\div\left(-\frac{6}{5}\right)^2-\left[\frac{1}{5}+\frac{4}{25}+\left(-\frac{4}{5}\right)^2\right]\div\left(-\frac{6}{5}\right)-\frac{12}{25}$ $\left[\frac{1}{50}\right]$

68.** $\frac{5}{30}-\frac{1}{7}\times\left[\left(4-\frac{18}{27}\right)\div\frac{8}{3}+\left(\frac{3}{8}\right)^2\times\left(\frac{7}{4}\times\frac{7}{9}+\frac{5}{12}\right)\right]$ $\left[-\frac{1}{21}\right]$

69.** $-\frac{7}{3}+\left[\left(\frac{17}{34}+\frac{1}{3}\right)\left(\frac{6}{5}\right)^2-\left(\frac{6}{5}-\frac{9}{12}\right)\left(\frac{4}{3}\right)\right]\div\frac{9}{5}$ $\left[-2\right]$

70.** $\left\{\frac{15}{12}-\frac{10}{6}\times\left[\frac{5}{14}+\frac{5}{21}-\frac{1}{30}\div\left(1-\frac{13}{20}\right)\right]^2-1\right\}\times\frac{16}{5}$ $\left[-\frac{8}{15}\right]$

71.** $-\left(1-\frac{1}{5}\right)+\left[\frac{1}{2}+\left(2-\frac{7}{4}\right)^2\times\left(\frac{11}{3}+\frac{7}{5}\right)\right]\div\left(-\frac{7}{21}-\frac{16}{12}+3\right)$ $\left[-\frac{3}{16}\right]$

72.** $\left[-\frac{16}{3}\times\left(2+\frac{3}{5}\right)\times\left(-\frac{1}{2}\right)^3-\left(\frac{1}{2}-\frac{1}{3}\right)\right]\times\frac{21}{3}\times\left(-\frac{2}{47}\right)\div\left(\frac{2}{3}-2\right)$ $\left[\frac{7}{20}\right]$

73.** $1+\left\{\left[\left(\frac{5}{4}+\frac{1}{3}\right)^2-\left(\frac{10}{8}-\frac{7}{21}\right)^2\right]\div\left(1+\frac{2}{3}\right)\right\}\times\frac{12}{14}$ $\left[\frac{13}{7}\right]$

74.** $\left\{\left(-\frac{3}{2}\right)^3\div\left[\left(-\frac{2}{3}-\frac{1}{15}\right)-\left(\frac{-3}{2}\right)^3-\left(\frac{2}{3}-\frac{7}{5}\right)\right]\right\}+\left(-1+\frac{2}{3}+\frac{3}{4}-\frac{8}{12}\right)$ $\left[-\frac{5}{4}\right]$

75.** $\left\{-\frac{5}{32}+\left[\frac{2}{3}+\frac{5}{6}\div\left(-\frac{5}{3}\right)\right]\times\left(-\frac{9}{16}\right)\right\}^2\div\left[\left(\frac{2}{5}+\frac{5}{12}-\frac{7}{60}\right)\times\left(-1+\frac{2}{7}\right)^2\right]$ $\left[\frac{7}{40}\right]$

76.** $\left(-\frac{3}{7}+1\right)\times\left\{-\frac{54}{8}\times\left[-\frac{2}{9}+\frac{3}{15}+\left(-\frac{7}{15}\right)\div\left(-2-\frac{3}{2}\right)\right]-\frac{19}{8}\right\}$ $\left[-\frac{25}{14}\right]$

77.** $\left\{\left(2-\frac{3}{2}\right)^2+\left[\left(\frac{1}{3}-\frac{2}{5}\right)\times\left(-\frac{1}{3}+\frac{3}{2}\right)+\frac{1}{30}\right]\div\frac{1}{9}\right\}\div\left[\left(\frac{2}{5}+\frac{7}{5}\times\frac{3}{7}\right)^2-\frac{9}{20}\right]$ $\left[-\frac{3}{11}\right]$

78.** $\frac{1}{6}+\left[\frac{5}{21}\div\left(1+\frac{7}{3}\right)+\left(-\frac{1}{2}-\frac{4}{7}\right)\times\left(\frac{2}{4}-\frac{4}{5}\right)\right]\div\left[-\frac{2}{5}\times\left(1-\frac{1}{4}\right)\right]$ $\left[-\frac{8}{7}\right]$

79.** $\left\{\left[\left(1+\frac{3}{10}\times\frac{25}{9}\right)^2-\frac{49}{36}\right]\times\frac{1}{7}+\frac{5}{7}\right\}\times\left[\left(\frac{5}{12}\times\frac{4}{5}\right)^3\div\frac{1}{27}+\frac{1}{3}\right]$ $\left[\frac{4}{3}\right]$

80.** $\left\{9\times\left[\left(\frac{1}{3}-\frac{2}{5}\right)\times\left(-\frac{1}{3}+\frac{3}{2}\right)+\frac{1}{30}\right]+\left(2-\frac{3}{2}\right)^2\right\}\div\left[\left(\frac{2}{5}+\frac{7}{5}\times\frac{3}{7}\right)^3-\frac{3}{10}-\frac{3}{20}\right]$ $\left[-\frac{3}{11}\right]$

81.** $\frac{28}{3}\times\left\{\frac{13}{8}-\frac{1}{7}\times\left[\left(4-\frac{2}{3}\right)\div\frac{8}{3}+\left(\frac{3}{8}\right)^2\times\left(\frac{7}{4}\times\frac{7}{9}+\frac{5}{12}\right)\right]\right\}$ $\left[\frac{79}{6}\right]$

82.** $\left[-\left(-\frac{1}{2}+\frac{4}{3}-1\right)^2\div\left(-\frac{1}{3}\right)^2-\frac{1}{2}\times\left(-1+\frac{1}{2}\right)^2\right]\div\left\{\left[\left(-1+\frac{1}{5}\right)^2-\frac{1}{25}\right]\div\left(-1-\frac{4}{5}\right)\right\}$ $\left[\frac{9}{8}\right]$

83.** $\left\{\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3+\left(-\frac{2}{3}\right)^2\right\}\times\left(\frac{5}{4}+\frac{1}{2}+\frac{1}{8}\right)\div\left(-\frac{5}{3}\right)^2$ $\left[\frac{1}{10}\right]$

84.** $\left\{\left(-2-\frac{3}{4}\right)^2\div\left[\left(\frac{7}{4}+\frac{1}{8}\right)^2\div\left(1+\frac{1}{6}-\frac{1}{3}\right)^2+\frac{5}{2}\right]\right\}\div\left[\left(\frac{3}{10}-\frac{9}{5}\right)\times\left(-\frac{2}{3}\right)\right]$ $\left[1\right]$

85.** $-\frac{5}{14}-\frac{5^2}{7^2}\div\left(2-\frac{3}{14}\right)+\left[\left(-1+\frac{1}{3}\right)^2+\left(-\frac{1}{2}\right)^3\div\left(\frac{5}{4}+1\right)\right]\times\frac{3^2}{7^2}$ $\left[-\frac{4}{7}\right]$

86.** $\left\{-\frac{3^3}{4}\times\left[-\frac{2}{9}+\frac{3}{15}+\left(-\frac{7}{15}\right)\div\left(-2-\frac{3}{2}\right)\right]-\frac{19}{8}\right\}\times\frac{4}{7}$ $\left[-\frac{25}{14}\right]$

87.** $\left\{\left(-\frac{2}{5}-\frac{4}{15}\right)\times\left[\frac{1}{4^2}-\left(-1+\frac{1}{4}-\frac{1}{2}\right)^2\right]\right\}\div\left[\left(9\times\frac{5}{18}\right)\div\left(\frac{3}{2}+1\right)^2\right]$ $\left[\frac{5}{2}\right]$

88.** $-\frac{4}{5}+\left[\frac{1}{2}+\left(2-\frac{7}{4}\right)^2\times\left(\frac{11}{3}+\frac{7}{5}\right)\right]\div\left(-\frac{30}{18}+3\right)$ $\left[-\frac{3}{16}\right]$

89.** $\left\{\left[\left(1+\frac{3}{10}\times\frac{25}{9}\right)^2-\frac{49}{36}\right]\times\frac{1}{7}+\frac{5}{7}\right\}\times\left[\left(\frac{5}{12}\times\frac{4}{5}\right)^3\div\frac{1}{27}+\frac{1}{3}\right]$ $\left[\frac{4}{3}\right]$

90.** $\left[\left(-\frac{5}{2}+\frac{5}{3}\right)^2-1+\frac{3}{12}\right]\div\left[-\frac{1}{11}\times\left(-\frac{11}{12}\right)^2+\left(-\frac{3}{4}+\frac{2}{3}\right)^2\right]-\left(-\frac{1}{4}\right)\times\left(-\frac{2}{5}\right)$ $\left[\frac{7}{10}\right]$

91.** $\left[-\frac{5}{3}\div\left(-\frac{2}{3}\right)^2+\frac{1}{8}\div\left(-\frac{3}{2}+\frac{5}{6}+\frac{1}{3}\right)^2\right]\times\left[\frac{1}{4}-\left(-\frac{2}{5}-\frac{1}{4}\right)\div\left(-2+\frac{3}{8}\right)-\frac{3}{12}\right]$ $\left[\frac{21}{20}\right]$

92.** $\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$ $\left[\frac{1}{10}\right]$

93.** $-\frac{1}{9}\times\left\{\frac{1}{2}-\left[-\frac{33}{32}\times\left(-\frac{7}{11}+1\right)^2\div\frac{1}{11}\right]^2\times\left(\frac{1}{3}+\frac{1}{9}\right)\right\}^2\div\left(\frac{1}{4}-\frac{1}{3}\right)$ $\left[\frac{1}{3}\right]$

94.** $\left\{\left[\frac{5}{12}\times\left(1-\frac{3}{15}\right)-\frac{37}{36}\right]\times\left(1+\frac{4}{5}\right)-\left(\frac{1}{18}+\frac{1}{72}\right)\right\}\div\left(2-\frac{7}{21}\right)$ $\left[-\frac{19}{24}\right]$

95.** $\dfrac{1+\frac{1}{3}\div\left(1+\frac{1}{18}-\frac{2}{12}\right)-\frac{1}{4}}{\frac{4}{9}\times\left(\frac{5}{14}-\frac{2}{3}\times\frac{3}{56}\right)+\frac{10}{28}}$ $\left[\frac{9}{4}\right]$

♣ $$\frac{\left[\left(\frac{15}{4}\times\frac{8}{45}\right)^3-\left(\frac{7}{8}\div\frac{14}{8}\right)^3\right]\div\left(\frac{4}{9}+\frac{2}{3}\times\frac{1}{2}+\frac{1}{4}\right)}{\left\{\left[\left(\frac{5}{8}-\frac{1}{8}\right)^4-\left(\frac{2}{7}\times\frac{14}{3}-1\right)^4\right]\div\left(\frac{1}{4}+\frac{1}{9}\right)\right\}\div\frac{15}{18}}$$ $\left[1\right]$

I.5 Arithmetic of square roots

For $a \geq 0$, we define $\sqrt{a}$ via the equation $(\sqrt{a})^2 = a$, and the requirement that $\sqrt{a} \geq 0$. The most famous square root of all is

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769 48\ldots$$

Very little is known about its digits, besides the fact that they will never terminate or repeat.

We have the basic identities

$$\sqrt{a}\,\sqrt{b} = \sqrt{a\,b} \qquad \sqrt{\frac{1}{a}} = \frac{1}{\sqrt{a}} \qquad a, b \geq 0 \tag{I.5.1}$$

from which we deduce

$$\sqrt{a^2} = \sqrt{a}\sqrt{a} = (\sqrt{a})^2 = a \qquad \sqrt{\frac{a}{b}} = \sqrt{a\,\frac{1}{b}} = \sqrt{a}\sqrt{\frac{1}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

The expression $\sqrt{a}$ is also called a *radical,* or a (quadratic) *surd,* while a is the *radicand.* An integer is *square-free* if it has no square divisors, that is, if all exponents in its prime factorization are equal to one. A convenient representation of a radical is obtained by extracting all squares, leaving a square-free kernel under the square root sign

$$\sqrt{125} = \sqrt{5^3} = 5\sqrt{5} \qquad \sqrt{84} = \sqrt{2^2 \cdot 3 \cdot 7} = 2\sqrt{21}.$$

Radicals can be removed from the denominator via *rationalization*

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{a\,b}{b^2}} = \frac{\sqrt{a\,b}}{b} \qquad \frac{1}{a+\sqrt{b}} = \frac{a-\sqrt{b}}{(a+\sqrt{b})(a-\sqrt{b})} = \frac{a-\sqrt{b}}{a^2-b}.$$

Thus

$$\frac{1}{\sqrt{120}} = \frac{\sqrt{120}}{120} = \frac{\sqrt{30}}{60} \qquad \sqrt{\frac{45}{28}} = \frac{3}{2}\sqrt{\frac{5}{7}} = \frac{3}{2}\frac{\sqrt{35}}{7} = \frac{3}{14}\sqrt{35}.$$

Examine carefully each step of the following rationalization

$$\begin{aligned}\frac{\sqrt{63}}{10-4\sqrt{7}} &= \frac{3\sqrt{7}}{2\,(5-2\sqrt{7})} = \frac{3\sqrt{7}\,(5+2\sqrt{7})}{2\,(5-2\sqrt{7})\,(5+2\sqrt{7})}\\ &= \frac{3\,(14+5\sqrt{7})}{2\left(5^2-(2\sqrt{7})^2\right)} = \frac{3\,(14+5\sqrt{7})}{2\,(25-28)} = -\frac{1}{2}\left(14+5\sqrt{7}\right).\end{aligned}$$

To estimate the size of an expression involving square roots, we sandwich it between two consecutive integers, and then square it

$$n \;<\; 3\sqrt{13} \;<\; n+1 \qquad \Longrightarrow \qquad n^2 \;<\; 9\cdot 13 \;<\; (n+1)^2$$

and since $10^2 = 100 < 117 < (10+1)^2 = 121$, we find $n = 10$. This gives $10 < 3\sqrt{13} < 11$, that is, 10 is the largest integer smaller than $3\sqrt{13}$, and 11 is the smallest integer larger than $3\sqrt{13}$. Suppose now we want to estimate the expression

$$\frac{29-3\sqrt{13}}{7}$$

which involves the same radical term $3\sqrt{13}$. From the above we have, changing sign, $-11 < -3\sqrt{13} < -10$, that is,

$$\frac{29-11}{7} < \frac{29-3\sqrt{13}}{7} < \frac{29-10}{7} \qquad \Longleftrightarrow \qquad 2+\frac{4}{7} < \frac{29-3\sqrt{13}}{7} < 2+\frac{5}{7}.$$

Simplify.

1. $\sqrt{5}$; $\sqrt{3^2}$; $\sqrt{8}$ — $\left[\sqrt{5};\quad 3;\quad 2\sqrt{2}\right]$

2. $\sqrt{(-7)^2}$; $-\sqrt{7^2}$; $\sqrt{(-27)\cdot(-3)}$ — $\left[7;\quad -7;\quad 9\right]$

3. $\sqrt{30}$; $\sqrt{80}$; $\sqrt{1000}$ — $\left[\sqrt{30};\quad 4\sqrt{5};\quad 10\sqrt{10}\right]$

4. $\left(\sqrt{2}\right)^3$; $\left(\sqrt{5}\right)^4$; $\left(\sqrt{7^3}\right)^5$ — $\left[2\sqrt{2};\quad 25;\quad 7^7\sqrt{7}\right]$

5. $\sqrt{63}$; $\sqrt{98}$; $\sqrt{108}$ — $\left[3\sqrt{7};\quad 7\sqrt{2};\quad 6\sqrt{3}\right]$

6. $\sqrt{60}$; $\sqrt{125}$; $\sqrt{243}$ — $\left[2\sqrt{15};\quad 5\sqrt{5};\quad 3^2\sqrt{3}\right]$

7. $\sqrt{3}\,\sqrt{7}$; $-\sqrt{8}\,\sqrt{2}$; $\sqrt{6}\sqrt{3}$ — $\left[\sqrt{21};\quad -4;\quad 3\sqrt{2}\right]$

8. $\sqrt{6}\,\sqrt{11}$; $\sqrt{6}\,\sqrt{10}$; $\sqrt{26}\,\sqrt{39}$ — $\left[\sqrt{66};\quad 2\sqrt{15};\quad 13\sqrt{6}\right]$

9. $\sqrt{3}\sqrt{11}$; $\sqrt{14}\sqrt{21}$; $\sqrt{15}\sqrt{65}$ — $\left[\sqrt{33};\quad 7\sqrt{6};\quad 5\sqrt{39}\right]$

10. $\left(\sqrt{6}\sqrt{12}\right)^3$; $\left(\sqrt{2}\sqrt{27}\right)^3$ — $\left[432\sqrt{2};\quad 162\sqrt{6}\right]$

11. $\sqrt{8}\,\sqrt{10}\,\sqrt{20}\,\sqrt{50}$; $\sqrt{2\sqrt{5}}\,\sqrt{5\sqrt{2}}\,\sqrt{\sqrt{10}}$ — $\left[200\,\sqrt{2};\quad 10\right]$

Simplify.

12. $4\sqrt{3}-5\sqrt{3}$; $-7\sqrt{5}+4\sqrt{5}+5\sqrt{5}$; $15\sqrt{2}+7\sqrt{2}-13\sqrt{2}$ — $\left[-\sqrt{3};\quad 2\sqrt{5};\quad 9\sqrt{2}\right]$

13. $6\sqrt{32}-8\sqrt{50}$; $6\sqrt{48}-7\sqrt{27}$; $\sqrt{128}-\sqrt{98}$ — $\left[-16\sqrt{2};\quad 3\sqrt{3};\quad \sqrt{2}\right]$

14. $2\sqrt{8}+5\sqrt{2}$; $4\sqrt{3}+6\sqrt{12}$; $\sqrt{108}+\sqrt{147}$ — $\left[9\sqrt{2};\quad 16\sqrt{3};\quad 13\sqrt{3}\right]$

15. $\sqrt{108}+\sqrt{147}$; $\sqrt{80}-\sqrt{45}$; $3\sqrt{125}+5\sqrt{20}-2\sqrt{45}$ — $\left[13\sqrt{3};\quad \sqrt{5};\quad 19\sqrt{5}\right]$

16. $\sqrt{2}-\sqrt{3}$; $\sqrt{8}-\sqrt{12}$; $\sqrt{160}+\sqrt{170}$ — $\left[\sqrt{2}-\sqrt{3};\quad 2\sqrt{2}-2\sqrt{3};\quad 4\sqrt{10}+\sqrt{170}\right]$

17.* $\sqrt{30}\,\sqrt{40}-\sqrt{3}-\sqrt{75}$ — $\left[14\sqrt{3}\right]$

18.* $\sqrt{21168}$ — $\left[84\sqrt{3}\right]$

Determine the largest integer smaller than the given expression.

19. $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$ $\left[1; \quad 1; \quad 2\right]$

20. $\sqrt{8}$; $\sqrt{11}$; $\sqrt{19}$ $\left[2; \quad 3; \quad 4\right]$

21. $\sqrt{35}$; $\sqrt{40}$; $\sqrt{45}$ $\left[5; \quad 6; \quad 6\right]$

22. $2\sqrt{2}$; $2\sqrt{3}$; $2\sqrt{7}$ $\left[2; \quad 3; \quad 5\right]$

23. $3\sqrt{6}$; $6\sqrt{2}$; $10\sqrt{2}$ $\left[7; \quad 8; \quad 14\right]$

24. $\sqrt{2}+3$; $20\sqrt{3}-5$; $\sqrt{122}-11$ $\left[4; \quad 29; \quad 0\right]$

25.** $7\sqrt{3}+1$; $3\sqrt{6}+7$; $2\sqrt{17}-1$ $\left[13; \quad 14; \quad 7\right]$

26.** $10\sqrt{2}+10$; $7\sqrt{2}\sqrt{3}$; $3\sqrt{13}-15$ $\left[24; \quad 17; \quad -5\right]$

27.** $\sqrt{17^2+17}$; $\sqrt{11^4-1}$; $\sqrt{100+\frac{1}{100}}$ $\left[17; \quad 120; \quad 10\right]$

Simplify, removing radicals at denominator.

28. $\frac{1}{\sqrt{2}}$; $\frac{3}{\sqrt{3}}$; $\frac{\sqrt{8}}{\sqrt{6}}$ $\left[\frac{\sqrt{2}}{2}; \quad \sqrt{3}; \quad \frac{2\sqrt{3}}{3}\right]$

29. $\sqrt{\frac{1}{5}}$; $\sqrt{\frac{2}{11}}$; $\sqrt{\frac{3}{5}}$ $\left[\frac{1}{5}\sqrt{5}; \quad \frac{1}{11}\sqrt{22}; \quad \frac{1}{5}\sqrt{15}\right]$

30. $\frac{\sqrt{3}}{\sqrt{40}}$; $\frac{\sqrt{5}}{\sqrt{18}}$; $\frac{\sqrt{11}}{\sqrt{20}}$ $\left[\frac{1}{20}\sqrt{30}; \quad \frac{\sqrt{10}}{6}; \quad \frac{1}{10}\sqrt{55}\right]$

31. $\sqrt{\frac{3}{48}}$; $\sqrt{\frac{12}{243}}$; $\sqrt{\frac{27}{75}}$ $\left[\frac{1}{4}; \quad \frac{2}{9}; \quad \frac{3}{5}\right]$

Simplify to the form $r\sqrt{d}$, where r is a fraction, and d is a square-free integer.

32. $\sqrt{11}-\sqrt{\frac{1}{11}}$; $6\sqrt{6}-9\frac{1}{\sqrt{6}}$ $\left[\frac{10}{11}\sqrt{11}; \quad \frac{9}{2}\sqrt{6}\right]$

33. $7\sqrt{2}+3\sqrt{\frac{1}{2}}$; $-3\sqrt{12}+\frac{\sqrt{6}}{\sqrt{2}}$ $\left[\frac{17}{2}\sqrt{2}; \quad -5\sqrt{3}\right]$

34. $5\sqrt{3}+7\sqrt{\frac{1}{3}}$; $-\sqrt{\frac{1}{8}}-\sqrt{\frac{1}{18}}$ $\left[\frac{22}{3}\sqrt{3}; \quad -\frac{5}{12}\sqrt{2}\right]$

35. $\sqrt{\frac{2}{7}}-\sqrt{\frac{1}{14}}$; $\frac{1}{\sqrt{20}}-\frac{1}{\sqrt{45}}$ $\left[\frac{\sqrt{14}}{14}; \quad \frac{\sqrt{5}}{30}\right]$

36. $-\sqrt{\frac{81}{6}}+4\sqrt{6}$; $\sqrt{\frac{49}{5}}-\sqrt{\frac{81}{5}}$ $\left[\frac{5}{2}\sqrt{6};\quad -\frac{2}{5}\sqrt{5}\right]$

37. $\sqrt{\frac{6}{7}}-\frac{\sqrt{7}}{\sqrt{6}}$; $\frac{\sqrt{5}}{\sqrt{21}}-\frac{\sqrt{3}}{\sqrt{35}}$ $\left[-\frac{\sqrt{42}}{42};\quad \frac{2}{105}\sqrt{105}\right]$

38. $\sqrt{\frac{7}{14}+\frac{1}{6}-\frac{5}{12}+1}$ $\left[\frac{1}{2}\sqrt{5}\right]$

39.* $\frac{5}{\sqrt{5}}-\frac{1}{\sqrt{45}}+\sqrt{\frac{49}{5}}$ $\left[\frac{7}{3}\sqrt{5}\right]$

40.* $\sqrt{\frac{7}{3}}-\sqrt{\frac{3}{7}}+\sqrt{\frac{36}{21}}$ $\left[\frac{10}{21}\sqrt{21}\right]$

41.* $\sqrt{\frac{2}{45}}-\frac{\sqrt{20}}{3\sqrt{8}}$ $\left[-\frac{\sqrt{10}}{10}\right]$

42.* $\sqrt{6}\sqrt{21}-\sqrt{\frac{32}{7}}$ $\left[\frac{17}{7}\sqrt{14}\right]$

Simplify to the form $r+s\sqrt{d}$, where r, s are fractions, and d is a square-free integer.

43. $(\sqrt{2}-\sqrt{3})^2$; $(\sqrt{8}+\sqrt{18})^2$ $\left[5-2\sqrt{6};\quad 50\right]$

44. $\sqrt{3}\,(\sqrt{3}-1)^2$; $(\sqrt{5}+\sqrt{6})^3$ $\left[-6+4\sqrt{3};\quad 23\sqrt{5}+21\sqrt{6}\right]$

45. $\frac{2+\sqrt{8}}{(\sqrt{2})^3}$; $\left(1-\frac{1}{\sqrt{5}}\right)(\sqrt{5}-1)^2$ $\left[1+\frac{\sqrt{2}}{2};\quad 8-\frac{16}{5}\sqrt{5}\right]$

46. $\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)^2$; $\left(\sqrt{60}-\frac{\sqrt{15}}{2}\right)^2$ $\left[\frac{1}{2};\quad \frac{135}{4}\right]$

47. $(1-\sqrt{2})(1+\sqrt{2})$; $\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)$ $\left[-1;\quad 1\right]$

48. $\frac{1}{(\sqrt{3}-1)^2}$; $\left(\frac{1}{3-\sqrt{2}}\right)^2$ $\left[1+\frac{1}{2}\sqrt{3};\quad \frac{11}{49}+\frac{6}{49}\sqrt{2}\right]$

49. $\frac{1+\sqrt{6}}{1-\sqrt{6}}$; $\frac{\sqrt{19}-17}{17-\sqrt{19}}$ $\left[-\frac{7}{5}-\frac{2}{5}\sqrt{6});\quad -1\right]$

Simplify.

50.** $(1+\sqrt{3})^5$ $\left[76+44\sqrt{3}\right]$

51.** $\dfrac{-\sqrt{3}+1}{\sqrt{3}-3}-\dfrac{1}{\sqrt{3}}$ $\Big[0\Big]$

52.** $\dfrac{1}{\sqrt{2}+\sqrt{3}}\,\sqrt{\dfrac{1}{2}}$ $\Big[\dfrac{\sqrt{6}-2}{2}\Big]$

53.** $\dfrac{1}{\sqrt{80}}-\dfrac{\sqrt{5}}{3\sqrt{5}-7}$ $\Big[\dfrac{15}{4}+\dfrac{9}{5}\sqrt{5}\Big]$

54.** $\dfrac{4-\sqrt{2}\sqrt{3}}{-\sqrt{9}+\sqrt{6}}$ $\Big[-\dfrac{6+\sqrt{6}}{3}\Big]$

55.** $\dfrac{-\sqrt{14}+\sqrt{18}}{\sqrt{8}\,(\sqrt{7}-2)}$ $\Big[\dfrac{-1+\sqrt{7}}{6}\Big]$

56.** $\dfrac{\sqrt{250}-\left(\sqrt{10}\right)^3}{\sqrt{2}\sqrt{5}-\sqrt{50}\sqrt{2}}$ $\Big[\dfrac{5(1+\sqrt{10})}{9}\Big]$

57.** $\dfrac{\sqrt{2}\sqrt{15}\sqrt{20}-(\sqrt{6})^3}{-5+2\sqrt{6}}$ $\Big[-48-20\sqrt{6}\Big]$

58.** $\dfrac{1}{\sqrt{5}}\,\dfrac{\sqrt{30}-\sqrt{12}\,\sqrt{15}}{(\sqrt{2}-\sqrt{3})^2}$ $\Big[-18-7\sqrt{6}\Big]$

59.** $\dfrac{-\sqrt{24}+\sqrt{6}}{(-3+\sqrt{6})^2}$ $\Big[-\dfrac{12+5\sqrt{6}}{3}\Big]$

60.** $\dfrac{(\sqrt{8})^3-\sqrt{8}}{\sqrt{8^3}+\sqrt{8^4}}$ $\Big[\dfrac{2\sqrt{2}-1}{8}\Big]$

61.** $\dfrac{-1}{7-3\sqrt{2}\sqrt{3}}\sqrt{\dfrac{2}{3}}$ $\Big[\dfrac{18+7\sqrt{6}}{15}\Big]$

62.** $\dfrac{\sqrt{2}\sqrt{6}}{5-\sqrt{3^3}}-\sqrt{\dfrac{1}{3}}$ $\Big[-\dfrac{27+16\sqrt{3}}{3}\Big]$

63.** $\left(\dfrac{1}{2\sqrt{2}-3}\right)^2-\dfrac{15}{\sqrt{50}}$ $\Big[17+\dfrac{21\sqrt{2}}{2}\Big]$

64.** $\sqrt{\dfrac{5}{27}}\,\dfrac{33}{2\sqrt{15}-7}$ $\Big[\dfrac{30+7\sqrt{15}}{3}\Big]$

65.** $\dfrac{1}{\sqrt{3}}\left(\dfrac{\sqrt{12}-\sqrt{21}}{\sqrt{7}-3}\right)$ $\Big[\dfrac{1+\sqrt{7}}{2}\Big]$

66.** $\left(\sqrt{20}-\sqrt{\frac{49}{5}}\right)\frac{4}{3+\sqrt{5}}$ $\left[-3+\frac{9}{5}\sqrt{5}\right]$

67.** $\frac{\sqrt{2\cdot 3^3}-\left(\sqrt{2\cdot 3}\right)^5}{\sqrt{2}\sqrt{3}-\sqrt{12}\sqrt{3}}$ $\left[\frac{33}{5}(1+\sqrt{6})\right]$

68.** $\sqrt{\frac{3}{5}}\left(\frac{1}{\sqrt{15}+3}\right)^2$ $\left[\frac{-15+4\sqrt{15}}{30}\right]$

69.** $\frac{30}{\sqrt{75}}-\frac{1}{(\sqrt{3}-2)^2}$ $\left[-7-2\sqrt{3}\right]$

70.** $\frac{21}{\sqrt{63}}-\frac{1}{5+2\sqrt{7}}$ $\left[\frac{5+\sqrt{7}}{3}\right]$

71.** $\frac{1}{\sqrt{28}}\left(\frac{\sqrt{63}-\sqrt{14}}{5\sqrt{2}-7}\right)$ $\left[\frac{11+8\sqrt{2}}{2}\right]$

72.** $\sqrt{\frac{1}{6}}-\frac{\sqrt{2}}{\sqrt{12}-\sqrt{18}}$ $\left[\frac{\sqrt{6}+2}{2}\right]$

73.** $\frac{1}{7+3\sqrt{5}}-\frac{30}{\sqrt{20}}$ $\left[\frac{7-15\sqrt{5}}{4}\right]$

74.** $\frac{(\sqrt{5})^3-\sqrt{5}}{(\sqrt{5})^5+(\sqrt{5})^6}$ $\left[-\frac{1}{25}+\frac{\sqrt{5}}{25}\right]$

75.** $\frac{7\frac{\sqrt{42}}{\sqrt{7}}+\frac{6\sqrt{18}}{\sqrt{3}}-2\sqrt{30}\sqrt{5}}{\sqrt{6}-3}$ $\left[-6-3\sqrt{6}\right]$

76.** $\frac{\frac{1}{5}\sqrt{1400}}{(\sqrt{7}+\sqrt{2})^2}-\frac{\sqrt{2}\sqrt{28}}{\frac{4\sqrt{21}}{\sqrt{6}}-9}$ $\left[\left(\frac{6}{5}\right)^2\sqrt{14}\right]$

77.** $\left(7\frac{\sqrt{42}}{\sqrt{7}}+\frac{6\sqrt{18}}{\sqrt{3}}-2\sqrt{30}\sqrt{5}\right)\frac{2-\sqrt{6}}{\sqrt{54}-9}$ $[2]$

♣ $\frac{(7-\sqrt{79})\,(3-\sqrt{10})^2\,(19+6\sqrt{10})}{(9-\sqrt{79})\,(8+\sqrt{79})}$ $[-1]$

♣ $\left(2+\sqrt{-1}\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(2-\sqrt{-1}\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(2+\sqrt{-1}\left(\frac{1-\sqrt{5}}{2}\right)\right)\left(2-\sqrt{-1}\left(\frac{1-\sqrt{5}}{2}\right)\right)$ $[29]$

♣ $(3+\sqrt{2})(5-4\sqrt{2})(5+2\sqrt{2})(9-7\sqrt{2})(5+\sqrt{2})(7-6\sqrt{2})(7+5\sqrt{2})+7\cdot 17\cdot 23$ $[0]$

I.6 Estimation

Numbers are typically written in base 10, using both positive and negative powers

$$x = 302.0574 \;=\; 3\times 10^2 + 0\times 10^1 + 2\times 10^0 + 0\times 10^{-1} + 5\times 10^{-2} + 7\times 10^{-3} + 4\times 10^{-4}.$$

From the above, we infer that

$$302 < x < 303 \qquad\qquad x \approx 302.$$

Expressions of this kind are *estimates,* which provide approximate information, stripped of irrelevant details. Whereas the expression $302 < x < 303$ is precise, the expression $x \approx 302$ is vague, yet meaningful in an appropriate context. Thus, if less accuracy is required, we may write $x \approx 300$.

Multiplication by a power of 10 amounts to shifting the decimal point

$$123.45 \times 10^{-4} = 0.012345 \qquad\qquad 123.45 \times 10^{4} = 1234500.$$

The *scientific notation* of a number $x \neq 0$ is given by $x = \pm a \times 10^k$, with k integer, and $1 \leq a < 10$

$$-0.00432 = -4.32 \times 10^{-3} \qquad\qquad 4100000 = 4.1 \times 10^{6}.$$

The simplest estimates occur with expressions where some terms are much smaller than others, and therefore can be neglected

$$4.15\times 10^2 + 3\times 10^{-5} = 415.00003 \approx 415 \qquad\qquad 4.15\times 10^9 + 3\times 10^2 = 4150000300 \approx 4150000000.$$

Here the quantities we neglect are 3×10^{-5} and 3×10^{2}; they give the same *relative* contribution, while being vastly different in magnitude.

If there are no obvious terms to neglect, we have to specify the accuracy. Let

$$x = 0.02799101\ldots$$

By an estimate of x with a given number n of *significant digits* of accuracy, we mean the following

n	estimate
1	$0.02 \leq x < 0.03$
2	$0.027 \leq x < 0.028$
3	$0.0279 \leq x < 0.0280$
4	$0.02799 \leq x < 0.02800$
5	$0.027991 \leq x < 0.027992$

When determining significant digits, the leading zeros are not counted. This is evident using scientific notation

$$\frac{123}{9990} = 0.0123\,123\,123\,123\,\ldots \approx 1.23\times 10^{-2} \approx 1.231231\times 10^{-2}.$$

The decimal representation of a number with a *finite* number of digits is non-unique. The simplest example is

$$1 = 1.0000000000\ldots = 0.9999999999\ldots$$

where we exploit the fact that a number with finitely many digits really has infinitely many trailing zeros.

$$30 = 30.000\ldots = 29.999\ldots$$
$$700100 = 700100.000\ldots = 700099.999\ldots$$
$$2.003 = 2.003000\ldots = 2.002999\ldots$$
$$10^{-3} = 0.001000\ldots = 0.000999\ldots$$

Subtracting a (relatively) small quantity from a number with finitely many decimals leads to a related phenomenon. Consider the following examples with great care

$$\begin{aligned}
&5 - 10^{-1} = 4.9\\
&5 - 10^{-7} = 4.9999999\\
&5 - 7\times 10^{-7} = 4.9999993\\
&8000000 - 10^5 = 7900000\\
&8000000 - 6\times 10^1 = 7999940\\
&10^{-3} - 10^{-4} = 0.0009\\
&10^{-3} - 10^{-6} = 0.000999
\end{aligned}$$

The coarsest estimates are *exponential:* they provide order of magnitude information. Let n be a non-negative integer, and consider the following inequalities

$$(i) \qquad 10^{n-1} \leq x < 10^n \qquad\qquad (ii) \qquad 10^{-n-1} \leq x < 10^{-n}.$$

In (i), the integer part of x has n decimal digits; alternatively, 10^{n-1} is the largest power of 10 smaller than x, and 10^n is the smallest power of 10 larger than x. In (ii), the fractional part of x has $n+1$ leading zeros, etc. In the following examples we have $n = 5$

$$10^4 < 12345.333 < 10^5 \qquad\qquad 10^{-6} < 0.00000997 < 10^{-5}.$$

A simple estimate of a fraction is obtained by computing its integer part (see section I.3). The value of a fraction increases by increasing the numerator or by decreasing the denominator, and decreases by decreasing the numerator or by increasing the denominator. If these increments are small compared with the size of numerator and denominator, then the change in the value of the fraction will also be small. For example, for large $m, n > 0$, all fractions below are very close to each other

$$\frac{m}{n+1} < \frac{m}{n} < \frac{m+1}{n} \qquad\qquad \frac{m-1}{n} < \frac{m}{n} < \frac{m}{n-1}. \tag{I.6.1}$$

To find out how close they are, we compute their difference. So, for instance

$$\frac{m+1}{n} - \frac{m}{n} = \frac{1}{n} \qquad\qquad \frac{m}{n} - \frac{m}{n+1} = \frac{m}{n^2+n}.$$

and in both cases one sees that the right-hand side is small if n is sufficiently large. (Think about it.)

Inequalities such as (I.6.1) are useful for estimation. For instance, let us estimate

$$x = \frac{14999}{901} \times \frac{45000}{5001}.$$

We find

$$\frac{14999}{901} < \frac{15000}{900} = \frac{50}{3} \qquad \frac{45000}{5001} < \frac{45000}{5000} = \frac{45}{5} = 9.$$

Thus

$$x < \frac{50}{3} \times 9 = 150, \qquad \text{and indeed} \qquad x = 149.9600\ldots \approx 150$$

The following estimates are often useful

$$2^{10} = 1024 \approx 10^3 \qquad \pi = 3.1415\ldots \approx \frac{22}{7} = 3.1428\ldots$$

and hence $\pi^2 \approx (22/7)^2 = (484/49) \approx 10$.

Write as a decimal number.

1. 0.81×10^0; $0.81/10^0$ — $[0.81; \quad 0.81]$
2. 1.1×10^2; 1.1×10^{-2} — $[110; \quad 0.011]$
3. 653×10^{-3}; 0.041×10^7 — $[0.653; \quad 410000]$
4. $37.501 + 10^{-2}$; $20000 + 9 \times 10^2$ — $[37.511; \quad 20900]$
5. $10^5 + 10^{-5}$; $2 \times 10^6 - 10^3$ — $[100000.00001; \quad 1999000]$
6. $6.999 + 10^{-3}$; $3.005 + 50 \times 10^{-4}$ — $[7; \quad 3.01]$
7. $\frac{123}{100}$; $\frac{123}{10000}$ — $[1.23; \quad 0.0123]$
8. $\frac{4000}{10^5}$; $\frac{3.3 \times 10^{-3}}{10^{-4}}$ — $[0.04; \quad 33]$

Write in scientific notation.

9. 1; 10; 0.1 — $[1 \times 10^0; \quad 1 \times 10^1; \quad 1 \times 10^{-1}]$
10. 300000; 0.003; 0.99 — $[3 \times 10^5; \quad 3 \times 10^{-3}; \quad 9.9 \times 10^{-1}]$
11. 12; 3.456; 0.00088 — $[1.2 \times 10^1; \quad 3.456 \times 10^0; \quad 8.8 \times 10^{-4}]$
12.* 10.33×10^3; 0.0074×10^{-3}; 50×10^5 — $[1.033 \times 10^4; \quad 7.4 \times 10^{-6}; \quad 5 \times 10^6]$

Sort in ascending order.

13. 0.2, 0.03, 0.004, 0.0005 $\left[0.0005 < 0.004 < 0.03 < 0.2\right]$

14. 1001001, 1000101, 1010001, 1000011 $\left[1000011 < 1000101 < 1001001 < 1010001\right]$

15. 0.1, 0.09, 0.11, 0.101 $\left[0.09 < 0.1 < 0.101 < 0.11\right]$

16.* 67.9, 67.909, 67.8999, 68.01 $\left[67.8999 < 67.9 < 67.909 < 68.01\right]$

Estimate x to the accuracy indicated in square brackets.

17. $x = 3 \times (33 \times 10^5 + 1)$ [2] $\left[9.9 \times 10^6 \leq x < 10^7\right]$

18. $x = 2 \times \dfrac{90}{60001} \times 10^4$ [2] $\left[29 \leq x < 30\right]$

19. $x = 4 \times 10^5 + 550 \times 10^3 + 501 \times 10^2$ [2] $\left[10^6 \leq x < 1.1 \times 10^6\right]$

20. $x = \dfrac{10^1 - 10^{-1}}{10^3}$ [2] $\left[9.9 \times 10^{-3} \leq x < 10^{-2}\right]$

21. $x = \dfrac{22 \times 10^{-3} + 20.02 \times 10^{-2}}{2}$ [2] $\left[0.11 \leq x < 0.12\right]$

22.* $x = 10^8 + 15 \times 10^6 + 85 \times 10^4$ [2] $\left[1.1 \times 10^8 \leq x < 1.2 \times 10^8\right]$

23.* $x = 1000 \times (10^{-1} + 2 \times 10^{-2} + 3 \times 10^{-3} + 4 \times 10^{-4})$ [3] $\left[123 \leq x < 124\right]$

24.* $x = 10^0 + 10^{-5} - 10^{-10}$ [1] $\left[1 \leq x < 2\right]$

25.** $x = 10^0 + 10^{-5} - 10^{-10}$ [7] $\left[1.000009 \leq x < 1.00001\right]$

Determine the largest power of 10, smaller than the given number.

26. 5; 12; 99 $\left[10^0; \quad 10^1; \quad 10^1\right]$

27. 101777; 9999; 1073741824 $\left[10^5; \quad 10^3; \quad 10^9\right]$

28. 101×102; 44×10^2; 8000×10^{-2} $\left[10^4; \quad 10^3; \quad 10^1\right]$

29. 2^4; 2^7; 2^{20} $\left[10^1; \quad 10^2; \quad 10^6\right]$

30. 60×170; 31^2; 32^2 $\left[10^4; \quad 10^2; \quad 10^3\right]$

31. $\dfrac{201}{2} \times 10$; $\dfrac{199}{2} \times 10$ $\left[10^3; \quad 10^2\right]$

32. $\dfrac{401}{20} \times \dfrac{501}{100}$; $\dfrac{8001}{20} \times \dfrac{51}{20}$ $\left[10^2; \quad 10^3\right]$

33.* $\frac{39001}{13} \times \frac{190}{570}$; $\frac{9999}{2000} \times \frac{1000}{51}$ $\left[10^3; \quad 10^1\right]$

34.* $\frac{40001}{20} \times \frac{1500}{2999}$; $\frac{59999}{30} \times \frac{300}{61}$ $\left[10^3; \quad 10^3\right]$

35.* $\frac{4499}{15} \times \frac{100}{3001}$; $\frac{3000+1}{1800} \times \frac{600}{10^2}$ $\left[10^0; \quad 10^1\right]$

36.** $\frac{42000+1}{70} \times \frac{1}{59999}$; $\frac{10^{-3} \times 999}{8001^2} \times 2^6$ $\left[10^{-2}; \quad 10^{-7}\right]$

37.** $\frac{6001}{15} \times \frac{800}{299}$; $10^4 \times \frac{14999}{300} \times \frac{1}{51}$ $\left[10^3; \quad 10^3\right]$

38.** $\left(2100 - 10^{-5}\right) \times \frac{20}{301} \times \frac{5}{7003}$ $\left[10^{-2}\right]$

39.** $\frac{1250001 \times 10^4}{50 \times 10} \times \frac{1}{499^2}$ $\left[10^2\right]$

40.** $\frac{2}{5} \times 25001 \times \frac{60}{5999}$ $\left[10^2\right]$

41.** $\left(9 + \frac{7}{10^7}\right) \times \frac{2000001}{180}$ $\left[10^5\right]$

42.** $10^{-2} \times \frac{30001}{42} \times \frac{7 \times 10^5}{499}$ $\left[10^4\right]$

43.** $\frac{96001}{600 \times 10^5} \times \left(\frac{1}{4 - 10^{-4}}\right)^2$ $\left[10^{-4}\right]$

44.** $\frac{42001}{2 \times 10^2} \times \frac{1}{6999} \times \frac{1}{3}$ $\left[10^{-2}\right]$

45.** $\left(24 - \frac{1}{10^4}\right) \times \frac{1}{12001} \times \frac{10^4}{2}$ $\left[10^0\right]$

Determine the largest power of 2, smaller than the given number.

46. 5; 12; 15 $\left[2^2; \quad 2^3; \quad 2^3\right]$

47. 31; 32; 33 $\left[2^4; \quad 2^4; \quad 2^5\right]$

48. 150; 200; 250 $\left[2^7; \quad 2^7; \quad 2^7\right]$

49. 300; 400; 500 $\left[2^8; \quad 2^8; \quad 2^8\right]$

50. 1000; 2000; 3000 $\left[2^9; \quad 2^{10}; \quad 2^{11}\right]$

51.** 10^3; $4 \cdot 10^3$; $5 \cdot 10^3$ $\left[2^9;\quad 2^{11};\quad 2^{12}\right]$

52.** 10^4; $5 \cdot 10^4$; 10^7 $\left[2^{13};\quad 2^{15};\quad 2^{23}\right]$

Determine the number of decimal digits in the integer part of the given number.

53. $\frac{12345}{2}$; $\frac{12345}{10}$; $\frac{12345}{1234}$ $\left[4;\quad 4;\quad 2\right]$

54. $\frac{19}{5} \times 300$; $\left(\frac{201}{2}\right)^2$; $\frac{1000^2}{3^2}$ $\left[4;\quad 5;\quad 6\right]$

55.** $\frac{10^5}{\pi^4} \times \frac{1}{10^{-2}}$ $\left[6\right]$

56.** $\left(\frac{41}{2}\right)^2 \times \frac{2^{20}}{10^7} \times \frac{5}{2}$ $\left[3\right]$

Part II: essential algebra

II.1 Monomials and polynomials

We begin with some basic identities

$$\begin{aligned}
x(y+z) &= xy + xz \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
x^2 - y^2 &= (x-y)\,(x+y) \\
x^3 - y^3 &= (x-y)\,(x^2 + xy + y^2)
\end{aligned} \tag{II.1.1}$$

from which we obtain

$$\begin{aligned}
(x-y)^2 &= (x+(-y))^2 = x^2 - 2xy + y^2 \\
(x-y)^3 &= (x+(-y))^3 = x^3 - 3x^2y + 3xy^2 - y^3 \\
x^3 + y^3 &= (x^3 - (-y)^3) = (x+y)\,(x^2 - xy + y^2) \\
x^4 - y^4 &= ((x^2)^2 - (y^2)^2) = (x^2 - y^2)(x^2 + y^2) = (x-y)(x+y)(x^2+y^2) \\
(x+y+z)^2 &= (x+(y+z))^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.
\end{aligned} \tag{II.1.2}$$

Given two polynomials $f(x)$ and $g(x)$, we can write

$$f(x) = g(x) \cdot q(x) + r(x)$$

where $q(x)$ and $r(x)$ are polynomials, and the degree of $r(x)$ is less than that of $g(x)$. The polynomials $q(x)$ and $r(x)$ are called the *quotient* and the *remainder*, respectively, of the division $f(x) \div g(x)$. They are computed with the long division algorithm.

Multiply.

1. $c^3\,c^5;\quad (x^3)^2;\quad (x\,y)^3$ $\left[c^8;\quad x^6;\quad x^3\,y^3\right]$

2. $(-x^2)^3;\quad (-x^3)^2;\quad -(x^3)^2$ $\left[-x^6;\quad x^6;\quad -x^6\right]$

3. $(-a^2b)(-3ab^2)(-7);\quad 4xy(-xy)(-xy^2)$ $\left[-21\,a^3b^3;\quad 4\,x^3y^4\right]$

4. $(a\,b^2\,c^3)^4;\quad (-aba^2b)^2(3bcbc)^3$ $\left[a^4\,b^8\,c^{12};\quad 27\,a^6\,b^{10}\,c^6\right]$

5. $\left(\frac{3}{4}xz\right)\left(-\frac{2}{9}x^2yz\right)\left(-\frac{12}{5}y^2z\right)$ $\left[\frac{2}{5}x^3y^3z^3\right]$

6. $\frac{\alpha}{2}\left(\frac{-\alpha\beta}{2}\right)^5 64\,(-\beta^2)$ $\left[\alpha^6\beta^7\right]$

Collect like terms.

7. $-1-3x^2-8x^2+4x+3-x+4x^2$ $\left[-7x^2+3x+2\right]$

8. $2a^5-3a^3+a^5-3a^3-a^5-2a^5+5a^3$ $\left[-a^3\right]$

9. $4ab+ab^2-2a-b^2+3ab-ab^2-7ab$ $\left[-2a-b^2\right]$

10. $-3x^2y-8x^2y+4xy^2+2-xy^2+4x^2y$ $\left[-7x^2y+3xy^2+2\right]$

11. $c^4d^2-c^3d+3c^2d-2c^4d^2+3c^3d-cd$ $\left[-c^4d^2+2c^3d+3c^2d-cd\right]$

12. $\frac{ab}{2}-b\frac{a}{4}-\frac{3}{2}ab$ $\left[-\frac{5}{4}ab\right]$

13. $-\frac{a^2b^2}{2}-b^2\frac{a^3}{4}-\frac{3}{4}(-ab)^2+2\frac{a^2b^2}{3}+a(ab)^2$ $\left[-\frac{7}{12}a^2b^2+\frac{3}{4}a^3b^2\right]$

14. $\frac{91}{26}a-\frac{9b}{15}+\frac{19}{38}+\frac{3}{5}a-\frac{a}{10}-1$ $\left[4a-\frac{3}{5}b-\frac{1}{2}\right]$

Expand, collecting like terms.

15. $(-a-b)^2;\quad (2x+y)^2$ $\left[a^2+2ab+b^2;\quad 4x^2+4xy+y^2\right]$

16. $(-a+b)^2;\quad -(x-3)^2$ $\left[a^2-2ab+b^2;\quad -x^2+6x-9\right]$

17. $(-\alpha\beta+1)^2;\quad (-6\theta+3\delta^2-\theta)^2$ $\left[\alpha^2\beta^2-2\alpha\beta+1;\quad 49\theta^2-42\theta\delta^2+9\delta^4\right]$

18. $3^2(2x-1)^2;\quad (ab^5+5)^2$ $\left[36x^2+9-36x;\quad a^2b^{10}+10ab^5+25\right]$

19. $(a^3+3b^2)(a^3-3b^2)$ $\left[a^6-9b^4\right]$

20. $(d^3+5)(d^3-5);\quad (7a^2+2)(2-7a^2)$ $\left[d^6-25;\quad 4-49a^4\right]$

21. $(-n-9\,m)(m\,3^2-n)$; $\quad -(-5a^5b^3+7c^6d^7)(5a^5b^3+7c^6d^7)$ $\quad\left[n^2-81m^2;\quad 25a^{10}b^6-49c^{12}d^{14}\right]$

22. $(x^2+x+1)(x^2+1-x)$ $\quad\left[x^4+x^2+1\right]$

23. $(x^2+x+1)^2$ $\quad\left[x^4+2\,x^3+3\,x^2+2\,x+1\right]$

24. $(-3x+y+5z)^2$ $\quad\left[9\,x^2+y^2+\,25z^2-6\,xy-30\,xz+10\,yz\right]$

25. $(5-2a)^3$ $\quad\left[125-150\,a+60\,a^2-8\,a^3\right]$

26. $(2m^3-4n)(3n+5m^2)$ $\quad\left[10m^5+6m^3n-20m^2n-12n^2\right]$

27. $(5+3t)(2t^4-3t^2-t-2)$ $\quad\left[6t^5+10t^4-9t^3-18t^2-11t-10\right]$

28. $(4z^2-3z^2+1)(-3z^4+5z^4-19-2z^4)$ $\quad\left[-19z^2-19\right]$

Compute (q,r), the quotient and remainder of polynomial division

29. $(x-1)\div(1-x)$; $\quad (x+1)\div(x-1)$ $\quad\left[(-1,0);\quad (1,2)\right]$

30. $(z-1)\div(z+1)$; $\quad (-3z+2)\div(z+2)$ $\quad\left[(1,-2);\quad (-3,8)\right]$

31. $(2b-1)\div(3b+1)$; $\quad (-7c+3)\div(3c+4)$ $\quad\left[\left(\frac{2}{3},-\frac{5}{3}\right);\quad \left(-\frac{7}{3},\frac{37}{3}\right)\right]$

32. $(a^2-1)\div(a+1)$; $\quad (a^2+1)\div(a+1)$ $\quad\left[(a-1,0);\quad (a-1,2)\right]$

33. $(x^2-7x+3)\div(x+2)$ $\quad\left[(x-9,\,21)\right]$

34. $(x^3+28)\div(x+3)$ $\quad\left[(x^2-3x+9,\,1)\right]$

35. $(-x^{10}+1)\div x^3$ $\quad\left[(-x^7,1)\right]$

36.* $(y^4-16y^2+3y)\div(-4+y)$ $\quad\left[(y^3+4y^2+3,\,12)\right]$

37.* $(-25-3k+k^2)\div(5+k)$ $\quad\left[(k-8,\,15)\right]$

38.* $(x^5+x^4+x^3+x^2+x+1)\div(x+1)$ $\quad\left[(x^4+x^2+1,0)\right]$

39.** $(3x^3-x-1)\div(2x-1)$ $\quad\left[\left(\frac{3}{2}\,x^2+\frac{3}{4}\,x-\frac{1}{8},-\frac{9}{8}\right)\right]$

40.** $(5x^3-x)\div(3x+1)$ $\quad\left[\left(\frac{5}{3}\,x^2-\frac{5}{9}\,x-\frac{4}{27},\frac{4}{27}\right)\right]$

Compute the remainder of polynomial division

41.** $(-x^4+3\,x^2+2\,x-1)\;\div\;(x^2+2)$ $\quad\left[2x-11\right]$

42.** $(x^4-x^3+1)\;\div\;(x^2-3)$ $\quad\left[-3x+10\right]$

43.** $(x^4 - 3x^3 + 2x + 1) \div (x^2 + 1)$ $\left[5x + 2\right]$

44.** $(z^4 - 3z) \div (-z^2 - z + 1)$ $\left[-6z + 2\right]$

45.** $(z^4 - 3z^3 + 2z - 4) \div (z^2 + 2)$ $\left[8z\right]$

46.** $(-x^4 - 7x^3 + 1) \div (-x^2 + 2)$ $\left[-3 - 14x\right]$

47.** $(3X^4 - 3X - 2) \div (-X + 2)$ $\left[40\right]$

48.** $(-2x^4 - 7x^3 + 2) \div (x + 3)$ $\left[29\right]$

49.** $(4w^5 + w^4 - 5) \div (-w + 1)$ $\left[0\right]$

50.** $(2a^4 - 5a - 2) \div (-a^2 + 2)$ $\left[6 - 5a\right]$

51.** $(3X^4 - 3X - 2) \div (-X + 2)$ $\left[40\right]$

Compute the quotient of polynomial division

52.** $(x^4 - 4x + 1) \div (x - 2)$ $\left[x^3 + 2x^2 + 4x + 4\right]$

53.** $(x^4 - x + 1) \div (-x + 3)$ $\left[-x^3 - 3\,x^2 - 9\,x - 26\right]$

54.** $(-y^4 - y^3 + 1) \div (y + 2)$ $\left[-y^3 + y^2 - 2y + 4\right]$

55.** $(3a^6 + 5a^4) \div (a^3 - 3)$ $\left[3a^3 + 5a + 9\right]$

56.** $(-2x^4 + 9x^2 + 2) \div (x - 2)$ $\left[-2x^3 - 4x^2 + x + 2\right]$

57.** $(4y^5 - y^4 + y^2 + 1) \div (y^3 + y^2 - 3)$ $\left[4y^2 - 5y + 5\right]$

58.** $(Z^4 - 2Z^3 - 6Z^2 + 1) \div (-Z + 3)$ $\left[-Z^3 - Z^2 + 3Z + 9\right]$

59.** $(z^6 - z^3 - 1) \div (z^3 - 3z)$ $\left[z^3 + 3z - 1\right]$

60.** $(a^4 - a^3 + a + 1) \div (-a - 2)$ $\left[-a^3 + 3a^2 - 6a + 11\right]$

61.** $(c^4 - c^3 + c + 1) \div (-c + 3)$ $\left[-c^3 - 2c^2 - 6c - 19\right]$

62.** $(2X^5 - X^4 + 2) \div (X^3 - 3X + 1)$ $\left[2X^2 - X + 6\right]$

63.** $(2a^4 - 3a^3 - 1) \div (-a + 1)$ $\left[-2a^3 + a^2 + a + 1\right]$

64.** $(c^4 - c^3 + c + 1) \div (-c + 3)$ $\left[-c^3 - 2c^2 - 6c - 19\right]$

65.** $(2a^4 - 3a^3 - 1) \div (-a + 1)$ $\left[-2a^3 + a^2 + a + 1\right]$

II.2 Polynomial factorization

We wish to express a polynomial as a product of polynomials of lower degree. We consider the identities (II.1.1–2) displayed at the beginning of the previous section. The first identity gives us the simplest tool for factorization: collecting

$$6r^2s - 8rs^2 = 2rs(3r - 4s).$$

Certain expressions with four terms can be factored by *grouping* (collecting twice)

$$xy + xz + wy + wz = x(y + z) + w(y + z) = (x + w)(y + z).$$

Thus

$$6a^2 - 14ab + 15a - 35b = 2a(3a - 7b) + 5(3a - 7b) = (2a + 5)(3a - 7b).$$

When grouping is possible, there are always two ways of doing it, by collecting terms in a different order

$$6a^2 - 14ab + 15a - 35b = 3a(2a + 5) - 7b(2a + 5) = (3a - 7b)(2a + 5).$$

Next we exploit the formula for the difference of two cubes

$$\begin{aligned} 3a^3b^2 - 24b^8 &= 3b^2(a^3 - 8b^6) = 3b^2(a^3 - (2b^2)^3) \\ &= 3b^2(a - 2b^2)(a^2 + 2ab^2 + 4b^4). \end{aligned}$$

Certain quadratic polynomials can be factored using the identity

$$ax^2 + (ac + b)x + bc = (ax + b)(x + c)$$

for which one must examine all possible divisors of the constant term bc. For instance, to factor the polynomial $x^2 - 5x - 6$ (corresponding to $a = 1$), we consider all divisors of $bc = -6$

$$-6 = (-1)(6) = (1)(-6) = (-2)(3) = (2)(-3).$$

Letting $b = 1$, $c = -6$, we find $ac + b = (1)(1) + (-6) = -5$, which is the coefficient of the term of degree 1, giving the factorization $x^2 - 5x - 6 = (x + 1)(x - 6)$. By contrast, the polynomial $x^2 - 4x - 6$ has no such factorization in polynomials with integer coefficients.

Factorization gran finale: check each step carefully

$$\begin{aligned} a^{25} - 2a^{13}b^{12} + ab^{24} &= a\left(a^{24} - 2a^{12}b^{12} + b^{24}\right) \\ &= a\left(a^{12} - b^{12}\right)^2 = a(a^6 - b^6)^2(a^6 + b^6)^2 \\ &= a(a^3 - b^3)^2(a^3 + b^3)^2(a^2 + b^2)^2(a^4 - a^2b^2 + b^4)^2 \\ &= a(a - b)^2(a^2 + ab + b^2)^2(a + b)^2(a^2 - ab + b^2)^2(a^2 + b^2)^2(a^4 - a^2b^2 + b^4)^2. \end{aligned}$$

Factor, by collecting.

1. $3+6x; \quad 7x^2-7x; \quad -8x+6x^2$ $\left[3(1+2x); \quad 7x(x-1); \quad -2x(4-3x)\right]$

2. $x^2-xy+3x; \quad -z^5+z^6+4z^4$ $\left[x(x-y+3); \quad z^4(z^2-z+4)\right]$

3. $-7ab-2b^2; \quad ab^3c-c(ba)^2-b^2c$ $\left[-b(7a+2b); \quad b^2c\,(ab-a^2-1)\right]$

4. $abcde-abce^2; \quad x^2y^4z^6-x^2y^4z^7$ $\left[abce(d-e); \quad (xy^2z^3)^2(1-z)\right]$

5. $35t^5-21t^4+14t^3-28t$ $\left[7t(5t^4-3t^3+2t^2-4)\right]$

6. $yx^2-yx^4+yx^6-yx^8$ $\left[yx^2(1-x^2+x^4-x^6)\right]$

7. $4x^8+2x^4-x^9-6x^5-x^9-2x^4$ $\left[-2x^5(x^4-2x^3+3)\right]$

8. $z^{40}+z^{20}+z^{30}+z^{40}-2z^{20}$ $\left[z^{20}(-1+z^{10}+2z^{20})\right]$

9. $(2a^2b)^2-14ab^2-(18b)a^2b$ $\left[2ab^2(2a^3-9a-7)\right]$

10. $abcd-bcde-cdef+defg$ $\left[d\,(abc-bce-cef+efg)\right]$

Factor, by grouping.

11. $x^2+3x+5x+15$ $\left[(x+5)(x+3)\right]$

12. $2\,\theta^2+22\,\theta+3\,\theta+33$ $\left[(\theta+11)(2\theta+3)\right]$

13. $x^2-7x-4x+28$ $\left[(x-4)(x-7)\right]$

14. $56\,z^2-24\,z+35\,z-15$ $\left[(8z+5)(7z-3)\right]$

15. $6\,x^2-14\,x+15\,x-35$ $\left[(2x+5)(3x-7)\right]$

16.* $26\,x^2-20\,x+39\,x-30$ $\left[(2x+3)(13x-10)\right]$

17.* $25\,x^2-10\,x-35\,x+14$ $\left[(5x-2)(5x-7)\right]$

18.* $15\,x^6+14\,x^3-30\,x^3-28$ $\left[(15x^3+14)(x^3-2)\right]$

Factor, by examining the constant term.

19. $x^2-3x+2; \quad z^2-2-z$ $\left[(x-1)(x-2); \quad (z+1)(z-2)\right]$

20. $x^2+17x+70; \quad y^2-17y+70$ $\left[(x+7)(x+10); \quad (y-7)(y-10)\right]$

21. $x^2-3x-40; \quad x^2+6x-40$ $\left[(x+5)(x-8); \quad (x-4)(x+10)\right]$

22. $60-19y+y^2; \quad 14x+48+x^2$ $\left[(y-15)(y-4); \quad (x+6)(x+8)\right]$

23. $-b^2 - 5b + 24; \quad 2\alpha + 8 - \alpha^2$ $\quad \Big[-(b-3)(b+8); \quad -(\alpha+2)(\alpha-4)\Big]$

24. $t^2 - 5t - 6; \quad 2a^2 - 11a - 6$ $\quad \Big[(t+1)(t-6); \quad (2a+1)(a-6)\Big]$

25. $3x^2 - 4x + 1; \quad 5x^2 + x - 6$ $\quad \Big[(3x-1)(x-1); \quad (5x+6)(x-1)\Big]$

Factor, as a difference of squares.

26. $x^2 - 4z^2; \quad a^2b^4 - c^4d^6$ $\quad \Big[(x-2z)(x+2z); \quad (ab^2 - c^2d^3)(ab^2 + c^2d^3)\Big]$

27. $9t^{14} - 1; \quad 144p^6 - 625$ $\quad \Big[(3t^7+1)(3t^7-1); \quad (12p^3+25)(12p^3-25)\Big]$

28. $98k^2 - 72; \quad 27x^2 - 300$ $\quad \Big[2(7k+6)(7k-6); \quad 3(3x+10)(3x-10)\Big]$

29. $162 - 2c^2; \quad 180p^4 - 20$ $\quad \Big[2(9+c)(9-c); \quad 20(3p^2+1)(3p^2-1)\Big]$

30. $x^3 - 9xy^2; \quad 12ab^3 - 3ab$ $\quad \Big[x(x-3y)(x+3y); \quad 3ab(2b-1)(2b+1)\Big]$

Factor, as the square/cube of a binomial.

31. $x^2 - 10x + 25; \quad 4x^2 - 4x + 1$ $\quad \Big[(x-5)^2; \quad (2x-1)^2\Big]$

32. $-t^2 - 40t - 400; \quad p^2 - 30p + 225$ $\quad \Big[-(t+20)^2; \quad (p-15)^2\Big]$

33. $s^4 + 6s^2 + 9; \quad -1 + 2t - t^2$ $\quad \Big[(s^2+3)^2; \quad -(t-1)^2\Big]$

34. $60x + 36 + 25x^2; \quad 169 + 36x^2 - 156x$ $\quad \Big[(5x+6)^2; \quad (6x-13)^2\Big]$

35. $-3z^3 + 6yz^2 - 3zy^2; \quad 98t^5 - 28t^4 + 2t^3$ $\quad \Big[-3z(z-y)^2; \quad 2t^3(7t-1)^2\Big]$

36. $x^3 + 6x^2 + 12x + 8$ $\quad \Big[(x+2)^3\Big]$

37. $a^3 - 3\,a^4 + 3\,a^5 - a^6$ $\quad \Big[a^3(1-a)^3\Big]$

38. $-8\,a^3 + 12\,a^2b - 6\,ab^2 + b^3$ $\quad \Big[(-2a+b)^3\Big]$

Factor

39.* $2\,x^4y^2 - 8\,x^3y^2 + 6\,x^2y^2$ $\quad \Big[2x^2y^2(x-1)(x-3)\Big]$

40.* $k^4 - 7k^2 - 4k^2 + 28$ $\quad \Big[(k-2)(k+2)(k^2-7)\Big]$

41.* $2a^7 - 2a^6 + a^3 - a$ $\quad \Big[a(a-1)(2a^5+a+1)\Big]$

42.* $c^3ab^4 - ba^4c^3$ $\quad \Big[abc^3\,(b-a)(b^2+ab+a^2)\Big]$

43.* $16x^4 - y^4$ $\quad \Big[(2x+y)(2x-y)(4x^2+y^2)\Big]$

44.* $z^3 + 125\,t^3$ $\quad \Big[(z+5t)(z^2-5zt+25\,t^2)\Big]$

45.** $a^6 - b^6$ $\left[(a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2)\right]$

46.** $48ca^5 - 27ab^4c$ $\left[3ac\,(4a^2+3b^2)(4a^2-3b^2)\right]$

47.** $77xy - 22x - 21y^2 + 6y$ $\left[(11x-3y)(7y-2)\right]$

48.** $abcd + bcde - cdef - adfc$ $\left[cd\,(a+e)(b-f)\right]$

49.** $25x^3 - x(3y)^2 + 50x^2 - 18y^2$ $\left[(x+2)(5x+3y)(5x-3y)\right]$

50.** $-66x^3y + 55x^3 + 6x^2y^2 - 5x^2y$ $\left[x^2(6y-5)(y-11x)\right]$

51.** $24k^3 + 81(mn)^3$ $\left[3(2k+3mn)(4k^2-6kmn+9m^2n^2)\right]$

52.** $27\,\theta^2 - 27\,\theta^2\gamma + 9\,\theta^2\gamma^2 - \theta^2\gamma^3$ $\left[\theta^2(3-\gamma)^3\right]$

53.** $9y^2 - x^2y^2 - 9z^2 + (xz)^2$ $\left[(z-y)(z+y)(x-3)(x+3)\right]$

54.** $-2x + y^4 + 18x^3 - 9x^2y^4$ $\left[(3x-1)(3x+1)(2x-y^4)\right]$

55.** $x^2y + 2x^2 - 9y - 18$ $\left[(y+2)(x+3)(x-3)\right]$

56.** $x^3 - 4x^2 - 9xy^2 + 36y^2$ $\left[(x-3y)(x+3y)(x-4)\right]$

57.** $-10(m^2n)^2 + 5m^3n^2 + 5m^4 - 10m^5$ $\left[5m^3(1-2m)(n^2+m)\right]$

58.** $7z - 1 - 28z^3 + (2z)^2$ $\left[(1+2z)(1-2z)(7z-1)\right]$

59.** $100r^2 - 25sr^2 - 4s^2 + s^3$ $\left[(4-s)(5r-s)(5r+s)\right]$

60.** $x - 7 - 4x^3 + 28x^2$ $\left[-(2x+1)(2x-1)(x-7)\right]$

61.** $-1 + (5x)^2 + 25x^2y^2 - y^2$ $\left[(1+y^2)(5x-1)(5x+1)\right]$

62.** $64a^2 - 16ba^2 - 4b^2 + b^3$ $\left[(4a-b)(4a+b)(4-b)\right]$

63.** $2b(5a)^2 + 45ab - 70ab^2 - 63b^2$ $\left[b(10a+9)(5a-7b)\right]$

64.** $xy^4 - x - (2y^2)^2 + 4$ $\left[(y-1)(y+1)(y^2+1)(x-4)\right]$

65.** $z^3 + 81u^2 - 9zu^2 - 9z^2$ $\left[(z-9)(z-3u)(z+3u)\right]$

66.** $3x^3 + 6x^2 - (xy)^2 - 2xy^2$ $\left[x(3x-y^2)(x+2)\right]$

67.** $15x^2 + 6x - 70x - 28$ $\left[(5x+2)(3x-14)\right]$

68.** $35a^5b + 21a^3b^2 - 10a^6 - 6a^4b$ $\left[a^3(7b-2a)(5a^2+3b)\right]$

69.** $2k^3 - 18km^2 - k^2 + 9m^2$ $\left[(2k-1)(k-3m)(k+3m)\right]$

70.** $3m^2p - 12n^2p - 8m^2q + 32n^2q$ $\Big[\,(m-2n)(m+2n)(3p-8q)\Big]$

71.** $50a^2 - 2b^4 - b^6 + (5ab)^2$ $\Big[\,(2+b^2)(5a+b^2)(5a-b^2)\Big]$

♣ $3\,x^{12} - 12288\,y^{12}$ $\Big[\,3\,(x-2\,y)(x^2+2\,xy+4\,y^2)(x+2\,y)(4\,y^2-2\,xy+x^2)(x^2+4\,y^2)(x^4-4\,x^2y^2+16\,y^4)\Big]$

II.3 Rational expressions

When manipulating rational expressions, keep numerator and denominator factored, whenever possible. Remember the rules for sign transfer

$$-\frac{a+b}{(a-c)(a-d)} = \frac{-(a+b)}{(a-c)(a-d)} = \frac{-a-b}{(a-c)(a-d)} = \frac{(a+b)}{(c-a)(a-d)} = \frac{(a+b)}{(a-c)(d-a)}$$

and the useful identity

$$\left(1-\frac{a^2}{b^2}\right) = \left(1+\frac{a}{b}\right)\left(1-\frac{a}{b}\right) = \frac{(b+a)(b-a)}{b^2}. \tag{II.3.1}$$

Simplify.

1. $-\dfrac{a-b}{-c-d}$; $\quad -\dfrac{1/a}{1/(b-a)}$ $\qquad \left[\dfrac{a-b}{c+d};\quad \dfrac{a-b}{a}\right]$

2. $\dfrac{a}{a^2}$; $\quad \dfrac{t^6}{t^9}$; $\quad \dfrac{z^7}{z^2}$ $\qquad \left[\dfrac{1}{a};\quad \dfrac{1}{t^3};\quad z^5\right]$

3. $\dfrac{t^{-9}}{t^{-3}}$; $\quad \dfrac{x^0}{x^{-2}}$; $\quad \dfrac{\theta^{-3}}{\theta^0}$ $\qquad \left[t^{-6};\quad x^2;\quad \theta^{-3}\right]$

4. $a\dfrac{a}{a^2}$; $\quad \left(-\dfrac{1}{a}\right)^2$; $\quad \left(\dfrac{a^2}{-b}\right)^5$ $\qquad \left[1;\quad \dfrac{1}{a^2};\quad -\dfrac{a^{10}}{b^5}\right]$

5. $z^{-2}\dfrac{z^5 w}{w^{-3}}$; $\quad \dfrac{2^2 x^6}{28xb}$ $\qquad \left[z^3w^4;\quad \dfrac{x^5}{7b}\right]$

6.** $\left(\dfrac{-a\,b^2c^3}{c\,b^3}\right)^3\left(\dfrac{a^5}{-b^7c}\right)^{-2}$ $\qquad \left[-\dfrac{b^{11}c^8}{a^7}\right]$

7.** $\left(\dfrac{x^2\,z^{-2}}{(1/y)^3}\right)\left(-\dfrac{y^2\,z}{x\,y^{-1}}\right)^{-3}$ $\qquad \left[-\dfrac{x^5}{z^5y^6}\right]$

8.** $\left(\dfrac{-a\,c^3}{bc^2a^3}\right)^{-3}\left(\dfrac{-c^7}{b^5\,(1/a)}\right)^2$ $\qquad \left[-\dfrac{c^{11}a^8}{b^7}\right]$

9.** $\dfrac{(-u^3v^{-2})^{-2}}{z}\left(\dfrac{-(1/z^2)\,v}{u^3}\right)^{-3}$ $\qquad \left[-u^3vz^5\right]$

10.** $\left(\dfrac{-1}{k\,n^2}\right)^3\left(m\,\dfrac{(-k/m^3)}{n}\right)^{-2}$ $\qquad \left[-\dfrac{m^4}{k^5n^4}\right]$

11.** $\left(\dfrac{n^3}{m\,(kn)^2}\right)^3\left(\dfrac{-(1/k)^2}{m^4n}\right)^{-2}$ $\qquad \left[\dfrac{n^5m^5}{k^2}\right]$

12.** $\left[\dfrac{XZ}{(1/Y^2)^2}\right]^{-3}\left[\dfrac{1}{X}\left(\dfrac{Y}{Z}\right)^3\right]^3$ $\qquad \left[\dfrac{1}{X^6Y^3Z^{12}}\right]$

13.** $\dfrac{(-1)^2}{A}\left(-B\left(-C^2\right)\right)^{-3}\dfrac{C^5}{(1/A^2)^2}$ $\left[\dfrac{A^3}{B^3C}\right]$

14.** $\left(\dfrac{XZ}{(-1/Y)^2}\right)^{-3}\left(\dfrac{1}{X}\left(\dfrac{Y}{Z}\right)^3\right)^2$ $\left[\dfrac{1}{X^5Z^9}\right]$

15.** $\left(\dfrac{Y/Z^2}{X\,Y^3}\right)^5\left(\dfrac{-\,(Y\,Z)^2}{X^{-4}}\right)^3$ $\left[-\dfrac{X^7}{Y^4Z^4}\right]$

16.** $\left(\dfrac{-\,(-1)^2}{(-1)^3}\right)^3\left(\dfrac{p^{-3}q^{-3}}{r^{-5}}\right)^2(1/q)^2$ $\left[\dfrac{r^{10}}{p^6q^8}\right]$

17.** $\left(\dfrac{(1/y)\,x^5}{-\,z^6}\right)^2\left(\dfrac{x}{z\,y\,y^{-3}}\right)^{-3}$ $\left[\dfrac{x^7}{z^9y^8}\right]$

18.** $\left(\dfrac{a}{b^4c\,a^{-1}}\left(-\dfrac{a^3}{bc^5}\right)^{-2}\right)^3$ $\left[\dfrac{c^{27}}{a^{12}b^6}\right]$

19.** $a^3\left(-\dfrac{a^2}{b\,(ca)^2}\right)^3\left(\dfrac{(1/c)^2}{b^4a}\right)^{-2}$ $\left[-\dfrac{a^5b^5}{c^2}\right]$

20.** $\left(\dfrac{(1/y)\,x^3}{-\,z^2}\right)^2\left(\dfrac{x}{z\,y^{-3}}\right)^{-3}$ $\left[\dfrac{x^3}{zy^{11}}\right]$

21.** $\dfrac{\left(-a^2b^{-3}\right)^{-2}}{c^2}\left(\dfrac{a}{(1/b^2)c}\right)^{-3}$ $\left[\dfrac{c}{a^7}\right]$

22.** $\dfrac{(-1)^2}{(-x)^{10}}\left(\dfrac{(-1/y)^3}{-\,x^{-3}}\right)^3$ $\left[\dfrac{1}{xy^9}\right]$

23.** $(1/f)^2\left(\dfrac{g^3}{h^2}\right)^2\left(-\dfrac{g}{h\,f^{-3}}\right)^{-3}$ $\left[-\dfrac{g^3}{hf^{11}}\right]$

24.** $\left(\dfrac{-\,1}{a^3}\right)^3\left[\left(\dfrac{b^2}{a}\right)^2\right]^{-2}\left(-a^{-2}b^3c^4\right)^2$ $\left[-\dfrac{c^8}{a^9b^2}\right]$

25.** $\left(\dfrac{X^3}{-\,Z^2}\right)^2\left(-\dfrac{Z^2}{Z\,Y^{-3}}\right)^{-3}(1/X)^2$ $\left[-\dfrac{X^4}{Z^7Y^9}\right]$

26.** $(1/a^{-1})^2\left(c^3b^{-2}\right)^2\left(-\dfrac{b}{c^3\,(1/a)}\right)^{-3}$ $\left[-\dfrac{c^{15}}{ab^7}\right]$

27.** $\left(\dfrac{1}{(-1)^3}\right)^2\left(\dfrac{-\,q^3}{(1/r)^2}\right)^3\left(\dfrac{-\,q^4}{(pq)^2r}\right)^{-1}$ $\left[q^7p^2r^7\right]$

Expand and simplify.

28. $\left(-u-\frac{1}{2t}\right)^2; \quad \left(\frac{1}{t}-\frac{1}{t^2}\right)^2$ $\left[u^2+\frac{u}{t}+\frac{1}{4t^2}; \quad \frac{1}{t^2}-\frac{2}{t^3}+\frac{1}{t^4}\right]$

29. $-\left(-\frac{1}{a^3}+\frac{1}{b^4}\right)^2; \quad \left(\frac{1}{8x}+\frac{8}{y}\right)^2$ $\left[-\frac{1}{a^6}+\frac{2}{a^3b^4}-\frac{1}{b^8}; \quad \frac{1}{64x^2}+\frac{2}{xy}+\frac{64}{y^2}\right]$

30. $\left(x-\frac{1}{x}\right)^2; \quad -\left[\beta^9-\left(\frac{1}{\beta}\right)^9\right]^2$ $\left[x^2-2+\frac{1}{x^2}; \quad -\beta^{18}+2-\frac{1}{\beta^{18}}\right]$

31. $\left(\beta-\frac{1}{\beta}\right)\left(\frac{\beta}{\beta^2}-\frac{\beta^3}{\beta^2}\right)$ $\left[2-\beta^2-\frac{1}{\beta^2}\right]$

32. $\left(\frac{1}{a}-\frac{a}{b^2}\right)\left(\frac{1}{a}+\frac{a}{b^2}\right)$ $\left[\frac{1}{a^2}-\frac{a^2}{b^4}\right]$

33. $\left(3r^2s+\frac{1}{7t}\right)\left(3r^2s-\frac{3t}{21\,t^2}\right)$ $\left[9r^4s^2-\frac{1}{49\,t^2}\right]$

34. $x^2\,(xy+4)\left(\frac{1}{xy}-3\right)\left(\frac{1}{x}\right)^2$ $\left[\frac{4}{xy}-3xy-11\right]$

Factor and simplify.

35. $\frac{7x-14y}{2y-x}; \quad \frac{a^6}{a^6b-a^6}$ $\left[-7; \quad \frac{1}{b-1}\right]$

36. $\frac{3t^3+12t^2+9t}{3t}$ $\left[t^2+4t+3\right]$

37. $\frac{2x^2}{6x^4-5x^3+2x^2}$ $\left[\frac{2}{6x^2-5x+2}\right]$

38. $\frac{-8a^4}{24a^7-32a^6-16a^4}$ $\left[\frac{1}{-3a^3+4a^2+2}\right]$

39. $\frac{-8x^5y^2+12x^4y^3+32x^2y^4}{4x^2y}$ $\left[y\,(-2x^3+3x^2y+8y^2)\right]$

40. $\frac{20x^4-25x^3+5x^2}{5x^2-5x}$ $\left[x(4x-1)\right]$

41. $\frac{-2-x}{x^2-4}; \quad \frac{16-y^6}{4-y^3}$ $\left[\frac{1}{2-x}; \quad 4+y^3\right]$

42. $\frac{\theta^2-2\theta+1}{\theta-1}; \quad \frac{q+3}{q^3+6q^2+9q}$ $\left[\theta-1; \quad \frac{1}{q(q+3)}\right]$

43. $\frac{a-b}{-a^3+2a^2b-ab^2}$ $\left[\frac{1}{a(b-a)}\right]$

44. $\left[\frac{x^4}{y^2} - \left(\frac{a}{b^2}\right)^2\right] \div \left(\frac{x^2}{y} + \frac{a}{b^2}\right)$ $\left[\frac{x^2}{y} - \frac{a}{b^2}\right]$

45. $\left(\frac{1}{9b^2} - \frac{18a^2}{32}\right) \div \left(\frac{1}{3b} - \frac{3a}{4}\right)$ $\left[\frac{1}{3b} + \frac{3a}{4}\right]$

46. $\frac{a^3b^2 - b^2}{b - ba}$ $\left[-b\left(a^2 + a + 1\right)\right]$

47. $\frac{\beta\theta^6 - \beta\theta^4}{(\beta\theta)^2 + \beta^2\theta}$ $\left[\frac{\theta^3(\theta - 1)}{\beta}\right]$

48. $\left(2u^2 + 6u\right)^3 \frac{1}{8u^3(u+3)^2}$ $\left[u + 3\right]$

49. $\frac{1}{e}\frac{e^7 - ef^6}{(-e^3 + f^3)(f^3 + e^3)}$ $\left[-1\right]$

Add/subtract and simplify.

50. $\frac{6}{x} - \frac{8}{x}$; $-\frac{13}{7x^2} + \frac{6}{7x^2}$ $\left[-\frac{2}{x}; \quad -\frac{1}{x^2}\right]$

51. $\frac{3x}{x-5} - \frac{15}{x-5}$; $\frac{6y^2}{x-3} + \frac{6}{3-x}y^2$ $\left[3; \quad 0\right]$

52. $\frac{5}{5x - 10} + \frac{x-3}{2-x}$ $\left[\frac{4-x}{x-2}\right]$

53. $\frac{3}{5t-1} + \frac{3}{5t}$ $\left[\frac{3(10t-1)}{5t(5t-1)}\right]$

54. $\frac{1}{1 - 2u + u^2} - \frac{1}{u-1}$ $\left[\frac{2-u}{(u-1)^2}\right]$

55. $\frac{(y-8)(y+2)}{y^2 - 11} - y\frac{8-y}{11-y^2} - 2\frac{8-y}{11-y^2}$ $\left[0\right]$

56. $\frac{7x+6}{4x-1} + \frac{5x-2}{1-4x}$ $\left[\frac{2x+8}{4x-1}\right]$

57. $\frac{c^2}{c-4} - \frac{16}{4-c}$ $\left[\frac{c^2+16}{c-4}\right]$

58. $\frac{k^2}{6-k} + 2\frac{18}{k-6}$ $\left[-k-6\right]$

59. $\frac{x^2}{x-2} - \frac{7x-10}{x-2}$ $\left[x - 5\right]$

60. $\dfrac{q}{q^2-9}-\dfrac{4}{4q-12}$ $\left[\dfrac{3}{(3+q)(3-q)}\right]$

61. $\dfrac{3x^2+8x-2}{x+6}-\dfrac{2x^2+3x+4}{x+6}$ $\left[x-1\right]$

62.* $\dfrac{2y-7}{y^2+8y+16}-\dfrac{3y-5}{y^2-16}$ $\left[\dfrac{-y^2-22y+48}{(y+4)^2(y-4)}\right]$

63.* $-a-b+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{b^2-1}{b}$ $\left[\dfrac{1-a^2}{a}\right]$

64. $\dfrac{10y}{2y^2-32}-\dfrac{2}{y}-\dfrac{3}{y-4}$ $\left[\dfrac{4(8-3y)}{y(y-4)(y+4)}\right]$

65. $\dfrac{2t}{2+t}+\dfrac{t}{t+3}+\dfrac{4}{t^2+5t+6}$ $\left[\dfrac{3t+2}{t+3}\right]$

66. $\dfrac{30}{4a^2-9}-\dfrac{5}{2a+3}+\dfrac{5}{3-2a}$ $\left[-\dfrac{10}{2a+3}\right]$

67.** $4\,\dfrac{1}{y+3}-\dfrac{y+7}{y^2+7y+12}-\dfrac{3}{y+4}$ $\left[0\right]$

68.** $\dfrac{1}{y^2-2y-15}+\dfrac{3}{y^2-10y+25}$ $\left[\dfrac{4y+4}{(y+3)(y-5)^2}\right]$

69.** $\dfrac{a^2}{a^2-4}-\dfrac{4}{a^2+2a}-\dfrac{a}{a-2}$ $\left[\dfrac{8-4a-2a^2}{a(a+2)(a-2)}\right]$

70.** $\dfrac{1}{y-3}-\dfrac{y+4}{2y^2-5y-3}$ $\left[\dfrac{1}{1+2y}\right]$

71.** $\dfrac{m^2}{m^3-m^2}-\dfrac{2}{m^2-7m+6}+\dfrac{m}{m-6}$ $\left[\dfrac{m^2-8}{m^2-7m+6}\right]$

72.** $\dfrac{x^2}{x^3-9x}-\dfrac{2x-3}{x^2+2x-15}$ $\left[\dfrac{-x^2+2x+9}{(x+5)(x-3)(x+3)}\right]$

73.** $\dfrac{-A^2}{A^2-3A-18}-\dfrac{1-A}{A+3}$ $\left[\dfrac{6-7A}{(A+3)(A-6)}\right]$

74.** $\dfrac{x}{x+1}-\dfrac{5}{x^2-3x-4}+\dfrac{x+1}{4-x}$ $\left[\dfrac{6}{4-x}\right]$

75.** $\dfrac{3}{x^2-x-6}+\dfrac{x}{x^2-3x-10}$ $\left[\dfrac{x^2-15}{(x-3)(x+2)(x-5)}\right]$

76.** $12\,\dfrac{1}{3z^2+5z+2}-4\,\dfrac{1}{2+3z}+\dfrac{2}{z+1}$ $\left[2\dfrac{z+6}{(2+3z)(z+1)}\right]$

77.** $\dfrac{k}{k^4-3k^3}-\dfrac{1}{k^3-6k^2+9k}$ $\left[-\dfrac{3}{k^2(k-3)^2}\right]$

78.** $\dfrac{4k}{k^2-6k+9}-\dfrac{3-k}{3k-9}$ $\left[\dfrac{1}{3}\left(\dfrac{k+3}{k-3}\right)^2\right]$

79.** $\dfrac{A}{2-A}-\dfrac{3A}{10+A^2-7A}$ $\left[\dfrac{A}{5-A}\right]$

80.** $\dfrac{k}{k^4-3k^3}-\dfrac{1}{k^3-6k^2+9k}$ $\left[-\dfrac{3}{k^2(k-3)^2}\right]$

81.** $\dfrac{2}{a^2-9}-\dfrac{3}{a^2+a-12}-\dfrac{1}{a+4}$ $\left[\dfrac{8-a-a^2}{(a+4)(a+3)(a-3)}\right]$

82.** $\dfrac{3}{x^2-9}-\dfrac{2}{x^2+x-12}$ $\left[\dfrac{x+6}{(x+4)(x-3)(x+3)}\right]$

83.** $\dfrac{30}{4z^2-9}-\dfrac{5}{2z+3}+\dfrac{5}{3-2z}$ $\left[-\dfrac{10}{2z+3}\right]$

84.** $\dfrac{-z}{z^2-5z}+\dfrac{2z}{z^2-3z-10}$ $\left[\dfrac{z-2}{(z+2)(z-5)}\right]$

85.** $\dfrac{3Y}{9+Y^2-6Y}-\dfrac{Y}{3-Y}$ $\left[\dfrac{Y^2}{(Y-3)^2}\right]$

86.** $\dfrac{2-14x}{1-14x+49x^2}-\dfrac{4-x^2}{7x-1}$ $\left[\dfrac{x^2-6}{7x-1}\right]$

87.** $\dfrac{y^4}{-16y^2+y^4}-\dfrac{y^2-1}{y^2-8y+16}$ $\left[\dfrac{-8y^2+y+4}{(y-4)^2(4+y)}\right]$

88.** $\dfrac{1}{x^2-4x+3}-\dfrac{2x+2}{1-x^2}$ $\left[\dfrac{2x-5}{(x-1)(x-3)}\right]$

89.** $\dfrac{4}{y-3}-\dfrac{y+7}{y^2-7y+12}-\dfrac{3}{y-4}$ $\left[-\dfrac{14}{(y-3)(y-4)}\right]$

90.** $\dfrac{2x}{x+2}+\dfrac{x}{x+3}-\dfrac{2x^2+4x-4}{x^2+5x+6}$ $\left[\dfrac{x+2}{x+3}\right]$

Simplify.

91. $\dfrac{\dfrac{1}{x}+2}{\dfrac{1}{3x}+1}$; $\dfrac{-2+\dfrac{1}{3a^4}}{\dfrac{2}{a^4}+1}$ $\left[\dfrac{3\,(1+2x)}{1+3x};\quad \dfrac{1-6a^4}{3\,(2+a^4)}\right]$

92. $\dfrac{\dfrac{a}{x}-\dfrac{a}{y}}{\dfrac{y-x}{a}}$; $\quad \dfrac{p}{2}\,\dfrac{-\dfrac{2}{p}+3}{3-\dfrac{p}{2}}$ $\left[\dfrac{a^2}{xy};\quad \dfrac{3p-2}{6-p}\right]$

93. $\dfrac{\dfrac{s^3}{s-t}}{\dfrac{s}{s^2-t^2}}$; $\quad \dfrac{\dfrac{8}{c}+c+9}{c+5+\dfrac{4}{c}}$ $\left[s^2(s+t);\quad \dfrac{c+8}{c+4}\right]$

94. $x\left(\dfrac{1}{x}-\dfrac{y}{x}\right)$; $\quad \left(\dfrac{3}{a^2}-\dfrac{3}{a^3}\right)a^2$ $\left[1-y;\quad \dfrac{3(a-1)}{a}\right]$

95.* $\dfrac{a^3+2a}{a+\dfrac{1}{a+\dfrac{1}{a}}}$ $\left[a^2+1\right]$

96. $\dfrac{(2a-b)^2+2b(3a-4b)-4a^2}{2b(1-b)+2b^2}$ $\left[a-\dfrac{7}{2}b\right]$

97. $\dfrac{5a^3b^3-2ab^2}{3a\left(2a^2b^3+b\right)+3ab\left(-2a^2b^2+1\right)}$ $\left[\dfrac{b}{6}\left(5a^2b-2\right)\right]$

98.** $\dfrac{(a-b)(a+b)\left(a^2+b^2\right)\left(a^4+b^4\right)}{a^8-b^8}$ $\left[1\right]$

99.** $\dfrac{(a^2-ab)(a^3+a^2b+ab^2+b^3)}{a^4-b^4}$ $\left[a\right]$

100.** $\dfrac{x^7-81\,x^3}{(x^2+3\,x)\,(x^2+9)}$ $\left[x^2(x-3)\right]$

101.** $-\dfrac{2}{3\,x^2}\left[\left(x^2-\dfrac{1}{3}y\right)^2-\dfrac{1}{9}y^2\right]^2-\dfrac{1}{9}x^2y\left(8x^2-\dfrac{8}{3}y\right)$ $\left[-\dfrac{2}{3}x^6\right]$

102.** $\left\{\left[\left(\dfrac{r}{2}+\dfrac{2s}{3}\right)^2-\left(\dfrac{1}{2}r-\dfrac{2}{3}s\right)^2\right]^2-2r^2s^2\right\}\dfrac{1}{2r^2}+\left(r-\dfrac{s}{3}\right)^2$ $\left[r^2-\dfrac{2}{3}rs\right]$

103.** $\left(\dfrac{x}{a}-\dfrac{1}{ax}\right)^2\left(-x-\dfrac{1}{x}\right)^2-\left(\dfrac{1}{x^4}-2\right)\dfrac{1}{a^2}$ $\left[\dfrac{x^4}{a^2}\right]$

104.** $\left[\left(\dfrac{1}{4}b-2a\right)\left(-\dfrac{1}{3}b\right)-\left(a+\dfrac{1}{2}b\right)\left(a-\dfrac{1}{2}b\right)+\left(a-\dfrac{1}{3}b\right)^2\right]\div\left(-\dfrac{1}{6}b\right)$ $\left[-\dfrac{5}{3}b\right]$

105.** $r\left[\dfrac{1}{2}rs-\left(\dfrac{1}{4}r+s\right)^2+\left(s+\dfrac{1}{4}r\right)\left(s-\dfrac{r}{4}\right)\right]+\left(s+\dfrac{1}{2}r\right)^3+s^2\left(-\dfrac{3}{2}r-s\right)$ $\left[\dfrac{3}{4}r^2s\right]$

106.** $\frac{3}{4}x\,\frac{x-y}{3}+\frac{1}{16}(x+y)^2-\left(\frac{3x-y}{4}-\frac{2x-y}{2}\right)^2-\left[\frac{1}{4}(x-y)^2+\frac{1}{2}xy\right]$ $\left[-\frac{1}{4}y^2\right]$

107.** $\left\{a\left[\left(\frac{a}{2}-b\right)\left(\frac{a}{2}+b\right)-\frac{2}{3}\left(a^2-b^2\right)\right]+\frac{5}{3}ab\right\}\div\left(-\frac{1}{3}a\right)$ $\left[\frac{5}{4}a^2+b^2-5b\right]$

108.** $9x^2-\left(\frac{1}{3}+3x\right)^2+\left[\frac{3}{4}x^2y-\frac{1}{2}x(x-xy)\right]\div\left(-\frac{1}{4}x\right)$ $\left[-\frac{1}{9}-5xy\right]$

109.** $\left(\frac{1}{2}x^2-1\right)^2+\left[\left(\frac{1}{2}x-\frac{3}{2}y\right)^2+y\left(\frac{3}{2}x-\frac{9}{4}y\right)+\frac{3}{2}x^2\right]^2\div\left(-\frac{7}{4}\right)$ $\left[-\frac{3}{2}x^4-x^2+1\right]$

110.** $(2x-3)^2-\left\{(x-3)^2-3\left[\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}-x\right)^2\right]\right\}$ $\left[3x^2\right]$

111.** $\left(\frac{a}{3}-\frac{b}{9}\right)\left[\frac{1}{4}\left(3a-\frac{2}{3}b\right)^2+\left(\frac{3}{2}a+\frac{1}{3}b\right)^2+\left(3a-\frac{2}{3}b\right)\left(\frac{3}{2}a+\frac{1}{3}b\right)\right]$ $\left[3a^3-a^2b\right]$

112.** $a\left[\frac{1}{2}ab-\left(\frac{1}{4}a+b\right)^2+\left(b+\frac{1}{4}a\right)\left(b-\frac{1}{4}a\right)\right]+\left(b+\frac{1}{2}a\right)^3+b^2\left(-\frac{3}{2}a-b\right)$ $\left[\frac{3}{4}a^2b\right]$

113.** $\left[\frac{2}{9}a^2b^2-\frac{2}{9}(3a^2-ab)^2+\frac{2}{3}a^4\right]^2+\frac{16}{9}a^4(2a^3b-a^4)$ $\left[\frac{16}{9}a^6b^2\right]$

114.** $\left[(x+2y)(x+y)-\frac{4x}{3}(4y-1-3y^2)-4x\left(y^2+\frac{1}{3}\right)-(x^2+2y^2)\right]\div\left(-\frac{7}{3}xy\right)$ $\left[1\right]$

115.** $\left\{x^2y+\frac{x}{2}\left[3x^2\left(\frac{x-2y}{x}\right)^2+(2x+y)^2-6x^2+6xy\right]\right\}\div(-5xy)$ $\left[-\frac{x^2+13y^2}{10y}\right]$

116.** $\left[y^2\left(\frac{3}{2}xy-\frac{1}{4}y\right)^2-\frac{1}{4}y^4\left(3x-\frac{1}{2}\right)^2+\frac{3}{4}xy^4\right]\div\left[\left(x-\frac{1}{2}y\right)^3+3xy\left(\frac{1}{2}x-\frac{1}{4}y\right)-x^3\right]$ $\left[-6xy\right]$

117.** $\left(-y+\frac{1}{2}x\right)^2-x\left[\left(2xy-\frac{1}{2}\right)\left(\frac{x}{2}+2y\right)-y(x+2y)^2+4y^3\right]$ $\left[\frac{1}{2}x^2+y^2\right]$

118.** $\left(\frac{1}{2}x-\frac{1}{3}y\right)^2\left[\left(\frac{3x}{2}+\frac{1}{2}\right)\left(\frac{3}{2}x-\frac{1}{2}\right)-\left(\frac{3}{2}x-\frac{1}{2}\right)^2-\frac{3x-1}{2}\right]$ $\left[0\right]$

119.** $a\left\{\frac{7}{9}ab^2-a\left[\left(\frac{1}{2}a-\frac{2}{3}b\right)^2+\frac{2}{3}ab\right]\right\}+\left(\frac{1}{2}a^2-\frac{1}{3}b^2\right)^2$ $\left[\frac{1}{9}b^4\right]$

120.** $y\left[\frac{1}{2}x\left(2x-\frac{4}{3}y\right)-\left(x+\frac{1}{3}y\right)^2\right]\div\left[\left(x-\frac{1}{3}y\right)\left(x+\frac{1}{3}y\right)-x^2\right]$ $\left[12x+y\right]$

121.** $(2a+1)^2+2\left\{\frac{(a-2)^2}{2}-\left[\left(a+\frac{1}{2}\right)^2+\left(\frac{1}{2}-a\right)^2\right]\right\}-(a+2)^2$ $\left[-4a\right]$

♣ $$\frac{b^2\left(\frac{3}{2}a\,b-\frac{1}{4}\,b\right)^2-\frac{1}{4}b^4\left(3\,a-\frac{1}{2}\right)^2+\frac{3}{4}a\,b^4}{\left(a-\frac{1}{2}\,b\right)^3+3\,a\,b\left(\frac{1}{2}\,a-\frac{1}{4}\,b\right)-a^3}$$ $\left[-6ab\right]$

II.4 Substitutions

A *substitution* in an algebraic expression $f(x)$ is the replacement of the indeterminate x with an expression E, for which we use the notation $x := E$ ('x becomes E'). This amounts to evaluating $f(E)$.

$$f(x) = \frac{1+3x}{x^2} \qquad x := -\frac{2}{a^2}.$$

To perform the substitution, first enclose every occurrence of x within parentheses

$$f(x) = \frac{1+3(x)}{(x)^2},$$

hence replace x by $-2/a^2$, and simplify:

$$f\left(-\frac{2}{a^2}\right) = \frac{1+3\left(-\frac{2}{a^2}\right)}{\left(-\frac{2}{a^2}\right)^2} = \frac{1-\frac{6}{a^2}}{\frac{4}{a^4}} = \frac{a^2-6}{a^2}\,\frac{a^4}{4} = \frac{1}{4}\,a^2(a^2-6).$$

The substitution

$$g(y) = \frac{7y^3+21y^4}{7y^5} \qquad y := -\frac{4}{2a^2}$$

is only apparently more complicated. Indeed, after simplifying

$$g(y) = \frac{7y^3(1+3y)}{7y^5} = \frac{1+3y}{y^2} \qquad y := -\frac{2}{a^2}$$

we find the previous example in disguise.

Simplify all expressions before substituting

Evaluate at the given values of the indeterminates.

1. $x^2 - 80y^2 \qquad \{x := 9,\ y := 1\}$ $\qquad [1]$

2. $p^2 + pq - q^2 \qquad \{p := 5,\ q := -4\}$ $\qquad [-11]$

3. $16 + x^3y - 3xy^2 - 126 \qquad \{x := -2,\ y := 5\}$ $\qquad [0]$

4. $\dfrac{38a-38b}{19a} \qquad \{a := 5,\ b := -2\}$ $\qquad \left[\frac{14}{5}\right]$

5. $\dfrac{a\,(2a-3b)}{c\,(3a-b)} \qquad \{a := -1,\ b := 2,\ c := 4\}$ $\qquad \left[-\frac{2}{5}\right]$

6. $\left(\frac{x+y\sqrt{5}}{2}\right)\left(\frac{x-y\sqrt{5}}{2}\right)$ $\{x := -7,\ y := -3\}$ $\Big[1\Big]$

7. $(a+(a+(a+b^2)^2)^2)^2$ $\{a := -1,\ b := 1\}$ $\Big[0\Big]$

8. $2x^5+1-x^3+x^4-3x^5$ $\{x := -1\}$ $\Big[4\Big]$

9. $\theta^2\left(-\frac{1}{\theta^3}+\frac{1}{\theta^4}-\frac{1}{\theta^5}\right)$ $\left\{\theta := \frac{1}{-q}\right\}$ $\Big[q(1+q+q^2)\Big]$

10. $\frac{s}{1+s}$ $\left\{s := \frac{2t}{t^4}\right\}$ $\Big[\frac{2}{t^3+2}\Big]$

11. $x\,\frac{x^2-y^3}{x^2y}$ $\{y := -x^2\}$ $\Big[-\frac{1+x^4}{x}\Big]$

12.* $\frac{a^2-14a+49}{7-a}$ $\left\{a := \frac{1}{b}\right\}$ $\Big[\frac{7b-1}{b}\Big]$

13.* x^2-y^2 $\left\{x := \frac{1+t^2}{1-t^2},\ y := \frac{2t}{1-t^2}\right\}$ $\Big[1\Big]$

♣ x^2-2y^2 $\{x := 22619537,\ y := 15994428\}$ $\Big[1\Big]$

Evaluate the expression in curly brackets.

14. $f(x) = -x$ $\{f(-a)\}$ $\Big[a\Big]$

15. $g(x) = 2x$ $\{g(3)\}$ $\Big[6\Big]$

16. $g(y) = y^2$ $\{g(-1/\sqrt{x})\}$ $\Big[\frac{1}{x}\Big]$

17. $F(z) = 1+z+z^2+z^3$ $\{F(-y)\}$ $\Big[1-y+y^2-y^3\Big]$

18. $f(x) = a^2-a-1$ $\{-f(b^{-1})\}$ $\Big[1+\frac{1}{b}-\frac{1}{b^2}\Big]$

19.* $G(y) = 1+y+y^2+y^3$ $\{G(-3k^2)\}$ $\Big[1-3k^2+9k^4-27k^6\Big]$

20.* $h(\theta) = -\frac{1}{\theta^3}-\frac{19}{8}$ $\{h(-2/3)\}$ $\Big[1\Big]$

21.* $F(X) = X-\frac{X}{X^2}$ $\left\{F\left(\frac{a}{a+1}\right)\right\}$ $\Big[-\frac{2a+1}{a(a+1)}\Big]$

22.** $f(b) = 6b-\frac{1}{3b^2}-\frac{1-b}{9b^3}$ $\{f(-1/(3Y))\}$ $\Big[3Y^3-2Y^2-\frac{2}{Y}\Big]$

23.** $L(y) = \dfrac{y}{3} - \dfrac{y^2 - y^3}{y}$ $\{L(-a^{-1})\}$ $\left[\dfrac{2a+3}{3a^2}\right]$

24.** $f(x) = -\dfrac{2x^2}{-x + 5x^{-1}}$ $\{f(-5/a^2)\}$ $\left[\dfrac{50}{a^2(a^4-5)}\right]$

25.** $g(x) = \dfrac{-3x^3 + 9x^2 - x}{x+5}$ $\{g(-1/(3z^2))\}$ $\left[\dfrac{1+9z^2+3z^4}{3z^4(15z^2-1)}\right]$

26.** $t(a) = \dfrac{2a^3 - a - 2}{2a^2 - 1}$ $\{t(-2b^{-2})\}$ $\left[\dfrac{2(b^6 - b^4 + 8)}{b^2(b^4-8)}\right]$

27.** $f(b) = \dfrac{27}{b^3} - \dfrac{2}{-b + (3/b)}$ $\{f(-3/X^2)\}$ $\left[\dfrac{X^2(2 + 3X^4 - X^8)}{X^4 - 3}\right]$

28.** $u(x) = \dfrac{x^2 - x - 1}{x - 1}$ $\left\{u\left(-\dfrac{1}{a^3}\right)\right\}$ $\left[\dfrac{-1 - a^3 + a^6}{a^3(a^3+1)}\right]$

29.** $f(z) = \dfrac{1}{\dfrac{2}{z} - 3z^2 - 2z^3}$ $\{f(-2/k)\}$ $\left[-\dfrac{k^3}{k^4 + 12k - 16}\right]$

30.** $h(x) = 1 - \dfrac{1}{x^2 - x + 1}$ $\{h(-1/(3a^3))\}$ $\left[\dfrac{1+3a^3}{1+3a^3+9a^6}\right]$

31.** $f(\beta) = \dfrac{1}{-\beta + \dfrac{1}{\beta^3 - 3}}$ $\{f(-x/y^2)\}$ $\left[\dfrac{y^2(x^3+3y^6)}{x^4 + 3xy^6 - y^8}\right]$

32.** $f(x) = \dfrac{x}{1-x^3} - \dfrac{1}{x^2}$ $\{f(-2/m^2)\}$ $\left[-\dfrac{m^4(16+m^6)}{4(8+m^6)}\right]$

33.** $g(x) = -x + \dfrac{1}{2}x^2 - \dfrac{1}{4}x^3$ $\{g(-2z^3)\}$ $\left[2z^3 + 2z^6 + 2z^9\right]$

34.** $r(x) = 1 - \dfrac{2}{1 - 2x - 4x^2}$ $\{r(-1/(2y))\}$ $\left[-\dfrac{y^2 - y + 1}{y^2 + y - 1}\right]$

35.** $f(x) = \dfrac{x}{x+1} - \dfrac{1}{x^2} - x$ $\{f(-1/\theta)\}$ $\left[\dfrac{-\theta^4 + \theta^3 - 1}{\theta(\theta - 1)}\right]$

36.** $q(x) = \dfrac{x^3}{x^2-1} - \dfrac{1}{x}$ $\{q(-1/\beta)\}$ $\left[\dfrac{\beta^4 - \beta^2 + 1}{\beta^3 - \beta}\right]$

37.** $f(z) = \dfrac{2z-1}{z} - \dfrac{2}{1 - \dfrac{1}{z^2}}$ $\{f(-k^2)\}$ $\left[\dfrac{k^4 - 2k^2 - 1}{k^2(k^4-1)}\right]$

38.** $f(k) = \dfrac{1}{k} + \dfrac{2}{k^2} - \dfrac{8k - k^3}{k^3}$ $\{f(-(2/X)^2)\}$ $\left[-\dfrac{3}{8}X^4 - \dfrac{1}{4}X^2 + 1\right]$

39.** $s(z) = \dfrac{z^3 - z - 1}{z - \dfrac{3}{z}}$ $\{s(-3/m)\}$ $\left[\dfrac{27 - 3m^2 + m^3}{m^2(3 - m^2)}\right]$

40.** $g(x) = -\dfrac{1}{x - x^2 - x^3}$ $\{g(-a^{-3})\}$ $\left[\dfrac{a^9}{a^6 + a^3 - 1}\right]$

41.** $f(x) = \dfrac{2}{x} - \dfrac{x}{x^2 - x^3}$ $\{f(1/s^2)\}$ $\left[\dfrac{s^2(2 - s^2)}{1 - s^2}\right]$

42.** $u(x) = -\dfrac{x}{1 + x - \dfrac{1}{x}}$ $\{u(-2/\theta)\}$ $\left[\dfrac{4}{\theta^2 + 2\theta - 4}\right]$

43.** $f(x) = \dfrac{1 + \dfrac{2}{x}}{x - \dfrac{4}{x}}$ $\{f(-2/a^2)\}$ $\left[-\dfrac{a^2}{2(a^2 + 1)}\right]$

44.** $L(s) = \dfrac{s^2 + s - 1}{s^2 + 1}$ $\left\{L\left(-\dfrac{1}{3}\right)\right\}$ $\left[-\dfrac{11}{10}\right]$

II.5 Algebraic expressions with square roots

In this course the expression $\sqrt{x}$ is defined only for $x \geq 0$. All square roots are assumed non-negative; thus

$$\sqrt{x^2} = \sqrt{x \cdot x} = |x|.$$

Consider the equation $\sqrt{x^3} = |x|\sqrt{x}$. Because all radicands are assumed to be non-negative, the absolute value is redundant here. However, its use is advisable, to improve clarity.

From the above, the multiplicative property of square roots (equation (I.5.1)), and the factorization (II.3.1), one derives the the following identities

$$\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 + \frac{a}{b}}\,\sqrt{1 - \frac{a}{b}} = \frac{\sqrt{b+a}\,\sqrt{b-a}}{|b|}.$$

For $x \neq 0$, we have

$$\frac{x}{|x|} = \frac{|x|}{x} = \mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

The function $\mathrm{sign}(x)$ is called the *sign function.* Thus $b = |b|\,\mathrm{sign}(b)$.

We have assumed that all radicands are non-negative. To see why this precaution is necessary, let us examine the identity $\sqrt{a}\sqrt{b} = \sqrt{ab}$ when a and b are negative . On the one hand, we have

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$$

while on the other

$$\sqrt{-2}\sqrt{-3} = \sqrt{-1}\,\sqrt{2}\,\sqrt{-1}\,\sqrt{3} = \left(\sqrt{-1}\right)^2\sqrt{2}\sqrt{3} = -\sqrt{6}.$$

The solution of this apparent paradox requires the use of complex numbers, which lie beyond the scope of this course†. Here just keep in mind that square roots are tricky, and exercise extra care when manipulating expressions where radicands may be negative.

Multiply, expanding the radicand.

1. $\sqrt{x}\,\sqrt{x-5}; \qquad \sqrt{7x}\,\sqrt{3x-4}$ $\qquad \left[\sqrt{x^2-5x}; \qquad \sqrt{21x^2-28x}\right]$

2. $\sqrt{x+3}\,\sqrt{x+2}; \qquad \sqrt{x+5}\,\sqrt{x-5}$ $\qquad \left[\sqrt{x^2+5x+6}; \qquad \sqrt{x^2-25}\right]$

3. $\sqrt{2}\,\sqrt{x^2-3}\,\sqrt{x^2+3}\,\sqrt{x^4+9}$ $\qquad \left[\sqrt{2x^8-162}\right]$

† The latter expression is the correct one, under any ‘sensible’ definition of the complex square root function.

Simplify.

4. $\sqrt{a}\sqrt{a}$; $\sqrt{b}\sqrt{b}\sqrt{b^2}$; $\sqrt{a^2 b}\,\sqrt{b^2 a}$ $\Big[a;\quad b^2;\quad |ab|\sqrt{ab}\Big]$

5. $\sqrt{a\,b^2}$; $\sqrt{y^2\,\theta^3}$; $\sqrt{12\,x^{12}}$ $\Big[|b|\sqrt{a};\quad |\,y\,\theta|\,\sqrt{\theta};\quad 2\,x^6\,\sqrt{3}\Big]$

6. $\sqrt{49z^2}$; $\sqrt{x^3-3x^2}$; $\sqrt{(x-3)^2}$ $\Big[7|z|;\quad |x|\sqrt{x-3};\quad |x-3|\Big]$

7. $\sqrt{x^2-4x+4}$; $\sqrt{36-12y+y^2}$ $\Big[|x-2|;\quad |y-6|\Big]$

8. $\sqrt{18+12z+2z^2}$; $\sqrt{49t+14t^2+t^3}$ $\Big[|z+3|\sqrt{2};\quad |7+t|\sqrt{t}\Big]$

9. $-\sqrt{t^2}-\sqrt{|t|^2}+2(\sqrt{|t|})^2$ $\Big[0\Big]$

10. $3|-z|-|2z|-\sqrt{36\,|z^3|}$ $\Big[|z|(1-6\sqrt{|z|})\Big]$

11. $\sqrt{t^5}+\sqrt{9t}-t^2\sqrt{t}+\sqrt{t^3}$ $\Big[\sqrt{t}(|t|+3)\Big]$

12. $\dfrac{\sqrt{z^4\,w}\,\sqrt{w}}{w^2}$; $\dfrac{\sqrt{s^4}\sqrt{t^2}}{t^2}$ $\Big[\dfrac{z^2}{w};\quad \dfrac{s^2}{|t|}\Big]$

13. $\dfrac{\sqrt{3+x}\,\sqrt{9-x^2}}{15+5x}$ $\Big[\dfrac{\sqrt{3-x}}{5}\Big]$

14. $\dfrac{(x-2\sqrt{2})^2-8}{x}$ $\Big[x-4\sqrt{2}\Big]$

15.** $\sqrt{a^3(a^2+3a)}-\sqrt{27+9a}$ $\Big[(a^2-3)\sqrt{3+a}\Big]$

16.** $\sqrt{x^2-x^3}-\sqrt{4-4x}$ $\Big[(|x|-2)\sqrt{1-x}\Big]$

Simplify, removing radicals at denominator.

17. $\dfrac{\sqrt{75\,x}}{\sqrt{3x}}$; $\dfrac{\sqrt{42\,k^3}}{\sqrt{7k}}$; $\dfrac{\sqrt{3q}}{\sqrt{27\,q^3}}$ $\Big[5;\quad |k|\sqrt{6};\quad \dfrac{1}{3|q|}\Big]$

18. $\dfrac{1}{\sqrt{a}}$; $\dfrac{1}{1+\sqrt{a}}$; $\dfrac{\sqrt{b}-1}{\sqrt{b}+1}$ $\Big[\dfrac{\sqrt{a}}{a};\quad \dfrac{1-\sqrt{a}}{1-a};\quad \dfrac{1+b-2\sqrt{b}}{b-1}\Big]$

19. $\dfrac{\sqrt{m}}{\sqrt{m}-\sqrt{n}}$; $-\dfrac{k}{\sqrt{2k}}+\dfrac{j}{\sqrt{9j}}$ $\Big[\dfrac{m+\sqrt{m\,n}}{m-n};\quad \dfrac{\sqrt{j}}{3}-\dfrac{\sqrt{2k}}{2}\Big]$

20. $\dfrac{|t|^3}{\sqrt{t^6}}-t\sqrt{\dfrac{125\,t}{5t^3}}$ $\Big[1-5\,\mathrm{sign}(t)\Big]$

21. $\dfrac{\sqrt{81x^2}-2(\sqrt{|x|})^2}{91x}$ $\Big[\dfrac{\mathrm{sign}(x)}{13}\Big]$

22. $\dfrac{2\sqrt{PQ}-P-Q}{\sqrt{Q}-\sqrt{P}}$ $\left[\sqrt{P}-\sqrt{Q}\right]$

23. $\dfrac{\sqrt{a^2+49b-14a\sqrt{b}}}{-7\sqrt{b}+a}$ $\left[\operatorname{sign}(a-7\sqrt{b})\right]$

24.** $\dfrac{\sqrt{a^2-4}-\sqrt{9a+18}}{\sqrt{8+4a}}$ $\left[\dfrac{\sqrt{a-2}-3}{2}\right]$

25.** $\sqrt{-b^2+\dfrac{9a^2}{b^4}}\sqrt{\dfrac{1}{3a+b^3}}$ $\left[\dfrac{\sqrt{3a-b^3}}{b^2}\right]$

26.** $\dfrac{\sqrt{a}\sqrt{b-(1/b)}}{\sqrt{ab^2+ab}}$ $\left[\dfrac{1}{b}\sqrt{b-1}\right]$

27.** $\dfrac{\sqrt{k^5-k^3}-\sqrt{1+(1/k)}}{\sqrt{k^4+k^3}}$ $\left[\dfrac{k^2\sqrt{k-1}-1}{k^2}\right]$

28.** $\dfrac{1}{\sqrt{k^3+k}}\sqrt{\dfrac{1}{k^4}-1}$ $\left[\dfrac{\sqrt{k(1-k^2)}}{k^3}\right]$

29.** $\dfrac{1}{\sqrt{2x-y^3}}\sqrt{\dfrac{4}{y^2}\left(\dfrac{x}{y}\right)^2-y^2}$ $\left[\dfrac{\sqrt{2x+y^3}}{y^2}\right]$

30.** $\dfrac{\sqrt{b^3-4b}}{\sqrt{b}}-\dfrac{\sqrt{4b+8}}{2}$ $\left[\sqrt{b+2}\ \left(\sqrt{b-2}-1\right)\right]$

31.** $\sqrt{x^2y-y^{-1}}\sqrt{1/(x^2y-x)}$ $\left[\dfrac{\sqrt{xy(xy+1)}}{xy}\right]$

32.** $\sqrt{a^{-1}-4a^3}\sqrt{(1+2a^2)^{-1}}$ $\left[\dfrac{\sqrt{a(1-2a^2)}}{a}\right]$

33.** $\dfrac{1}{x^2\sqrt{x+1}}\sqrt{2x^6-2x^4}$ $\left[\sqrt{2(x-1)}\right]$

34.** $\dfrac{2}{\sqrt{5a+1}}\sqrt{-a^2+\dfrac{1}{25}}$ $\left[\dfrac{2\sqrt{1-5a}}{5}\right]$

35.** $\dfrac{\sqrt{3+2t}}{2}\dfrac{1}{\sqrt{-t^2+9/4}}$ $\left[\dfrac{\sqrt{3-2t}}{3-2t}\right]$

36.** $\dfrac{\sqrt{\dfrac{x}{25}-x^3}}{3\sqrt{5x+1}}$ $\left[\dfrac{\sqrt{x}\sqrt{1-5x}}{15}\right]$

37.** $\dfrac{1}{\sqrt{4k^9 - k^5}}\sqrt{2k + \dfrac{1}{k}}$ $\left[\dfrac{\sqrt{2k^2-1}}{k^3(2k^2-1)}\right]$

38.** $\dfrac{z^2}{\sqrt{z^2 + z^4}}\sqrt{1 - \dfrac{1}{z^4}}$ $\left[\dfrac{\sqrt{z^2-1}}{|z|}\right]$

39.** $\dfrac{6a + 4b - \sqrt{96ab}}{\sqrt{3}\sqrt{4a} - \sqrt{8b}}$ $\left[\sqrt{3a} - \sqrt{2b}\right]$

40.** $\dfrac{1}{\sqrt{t^5 + 2t^2}}\sqrt{4t\dfrac{1}{t^2} - t^5}$ $\left[\dfrac{\sqrt{t\,(2-t^3)}}{t^2}\right]$

41.** $\dfrac{x^2}{\sqrt{7y - 6x^3}}\sqrt{\dfrac{49}{12}\left(\dfrac{y}{x^2}\right)^2 - 3x^2}$ $\left[\dfrac{\sqrt{3}}{6}\sqrt{7y + 6x^3}\right]$

42.** $\dfrac{\sqrt{a-1} - \sqrt{(2a)^2 - 4a}}{\sqrt{a^2 - a}}$ $\left[\dfrac{\sqrt{a} - 2a}{a}\right]$

43.** $\dfrac{\sqrt{t} - \left(\sqrt{t}\right)^3}{\sqrt{st - st^2}}$ $\left[\dfrac{\sqrt{s(1-t)}}{s}\right]$

44.** $\sqrt{a^{-1} - 4a^3}\sqrt{(1 + 2a^2)^{-1}}$ $\left[\dfrac{\sqrt{a(1-2a^2)}}{a}\right]$

45.** $\dfrac{1 + (2\sqrt{y} - 1)^3 - 8|y|\sqrt{y}}{6\sqrt{y}}$ $\left[1 - 2\sqrt{y}\right]$

46.** $\dfrac{1}{a + 2\,b}\left(\dfrac{a\sqrt{2ab}}{\sqrt{a}} + 2b\sqrt{2b}\right)$ $\left[\sqrt{2b}\right]$

II.6 Linear equations and inequalities

First we note some properties of equations and inequalities. For all a, b we have

$$\begin{array}{rcllll} a & = & b & \Longleftrightarrow & a \pm c = b \pm c & \text{all } c \\ a & = & b & \Longrightarrow & ac = bc & \text{all } c \\ & ac = bc & & \Longrightarrow & a = b & c \neq 0 \\ a & < & b & \Longrightarrow & ac < bc & c > 0 \\ a & < & b & \Longrightarrow & ac > bc & c < 0. \end{array}$$

The last two properties also hold if we replace strict inequalities with non-strict ones, that is, $<$ by $\leq$, etc.

A *linear equation* is an equation of the type $ax + b = 0$, where x is the unknown variable, while a and b are real numbers, with $a \neq 0$. Its solution is $x = -b/a$. A *linear inequality* takes the form $ax + b > 0$, $ax + b \geq 0$, etc. An equation/inequality is *indeterminate* if it is true for all values of the variable, e.g.,

$$3(7x - 5) + 5 \geq 21x + 10 \qquad \Longrightarrow \qquad 0 \geq 0.$$

An equation/inequality has *no solution* if it is false for all values of the variable, e.g.,

$$3x - x = 2x + 1 \qquad \Longrightarrow \qquad 0 = 1.$$

Next we consider systems of two linear equations. The method of solution is best illustrated with an example

$$\begin{cases} 5y = x + 3 \\ -x + y = 17. \end{cases} \tag{II.6.1}$$

We begin by solving the first equation. Given that x has unit coefficient, we choose to solve it for x, obtaining

$$x = 5y - 3. \tag{II.6.2}$$

Substituting this value of x into the second equation (II.6.1), we get

$$\begin{aligned} -(5y - 3) + y &= 17 \\ -5y + y &= 17 - 3 \\ -4y &= 14 \\ y &= -\frac{7}{2}. \end{aligned}$$

Now we substitute the above value of y in equation (II.6.2), to obtain

$$x = 5\left(-\frac{7}{2}\right) - 3 = -\frac{35}{2} - 3 = -\frac{41}{2}.$$

So the solution of the system of equations (II.6.1) is

$$x = -\frac{41}{2} \qquad y = -\frac{7}{2}.$$

Check that solving for y first gives the same result.

Solve.

1. $2 = \frac{1}{x}; \quad 2 = \frac{1}{x+1}; \quad \frac{1}{y} + \frac{3}{5y} = 4$ $\quad \left[\frac{1}{2}; \quad -\frac{1}{2}; \quad \frac{2}{5}\right]$

2. $\frac{z-9}{z-5} = \frac{1}{2}; \quad 5(x+2) - 3x = 2(x+7)$ $\quad \left[13; \quad \text{no solution}\right]$

3. $\frac{1}{2} - \frac{3}{2}x - \frac{2}{3} - \frac{5}{3}x = -\frac{19}{6}x - \frac{8}{12} + \frac{3}{6}$ $\quad \left[\text{indeterminate}\right]$

4. $\frac{4-10x}{3} + \frac{3x-1}{2} = \frac{5}{9} + \frac{8x-1}{4} - \frac{3}{4}$ $\quad \left[\frac{1}{3}\right]$

5. $\frac{2(x-5)}{5} - \frac{2(5x-1)}{7} = \frac{4(3x-5)}{5} - 4$ $\quad \left[\frac{11}{6}\right]$

6. $\frac{1}{x} + \frac{39}{65x} = 4$ $\quad \left[\frac{2}{5}\right]$

7. $\frac{2}{3x+15} = \frac{1}{2x}$ $\quad \left[15\right]$

8. $\frac{x-3}{x+5} = \frac{x+2}{x-4}$ $\quad \left[\frac{1}{7}\right]$

Reduce to a linear equation, and solve.

9. $x(x-3) = (x-1)(1+x)$ $\quad \left[\frac{1}{3}\right]$

10. $-3(1-z) + (z-2)^2 = -18 + (z+2)^2$ $\quad \left[3\right]$

11. $2(3-x) + (x+2)(x+3) = (x-2)(x-3)$ $\quad \left[-\frac{3}{4}\right]$

12. $\frac{(k+2)^2}{3} + \frac{(k-1)^2}{12} = \frac{(k-3)^2}{3} + \frac{4}{3} + \frac{(k+1)^2}{12}$ $\quad \left[1\right]$

13. $\left(\frac{1}{3} - x\right) 4x + 1 = 4x - \frac{8}{3} - 4x^2$ $\quad \left[\frac{11}{8}\right]$

Solve for the variable in curly brackets.

14. $7(\theta - x) = x - 3\theta \qquad \{x\}$ $\quad \left[\frac{5\,\theta}{4}\right]$

15. $ax^5 + bx^4 + cx^3 = 0 \qquad \{b\}$ $\quad \left[-\frac{ax^2+c}{x}\right]$

16.* $\frac{x}{y} - \frac{x^2}{y} = -\frac{7}{3y} + 4 \qquad \{y\}$ $\left[\frac{1}{4}\left(\frac{7}{3} + x - x^2\right)\right]$

17.* $\frac{a}{3}\left(ab^3 - c\right) - \frac{5a}{6} = 9a^2\left(\frac{b}{3}\right)^3 - c^2 \qquad \{a\}$ $\left[\frac{6c^2}{2c+5}\right]$

Solve.

18. $y + x = 3,\ -x + y = 1$ $\left[y = 2,\ x = 1\right]$

19. $2x = 3 - 2y,\ -x + 3y = 0$ $\left[y = \frac{3}{8},\ x = \frac{9}{8}\right]$

20. $2a + 3b = 4,\ 5a + 5b = 7$ $\left[b = \frac{6}{5},\ a = \frac{1}{5}\right]$

21. $11r - 9s = s - 1,\ 1 - 2r = 5r - 3s$ $\left[r = \frac{13}{37},\ s = \frac{18}{37}\right]$

22.* $x - 4y = 3(x + y) - 6,\ -9y = 4x - 5$ $\left[x = -\frac{19}{10},\ y = \frac{7}{5}\right]$

Solve.

23. $4x + 7 < -5; \quad x - 3 \leq 5; \quad -x - 4 < -8$ $\left[x < -3; \quad x \leq 8; \quad x > 4\right]$

24. $-\frac{4}{7}x < \frac{2}{21}; \quad -8x \geq \frac{16}{17}; \quad -\frac{6}{5}x < \frac{8}{15}$ $\left[x > -\frac{1}{6}; \quad x \leq -\frac{2}{17}; \quad x > -\frac{4}{9}\right]$

25. $\frac{1}{2}x + 4 - 3x > 5x - \frac{3}{2}$ $\left[x < \frac{11}{15}\right]$

26. $5(3x + 7) - 10x > 5(x + 8)$ [no solution]

27.* $3\xi + (\xi + 1)^2 > (\xi - 1)(\xi + 1) - 3\xi$ $\left[\xi > -\frac{1}{4}\right]$

28.* $2x - 5(3x + 2) > x - 1 + 2(x - 3)$ $\left[x < -\frac{3}{16}\right]$

29.* $(t - 4)(t + 4) - 2t > (t - 1)^2 - 3$ [no solution]

30.** $4\,\frac{z+1}{z^2 - 16} = \frac{1}{4 + z} - \frac{3}{4 - z}$ [no solution]

31.** $\frac{x}{2}(x - 3) - x\left(\frac{x}{2} - 4\right) - 5x - 2(x - 3) > 0$ $\left[x < \frac{4}{3}\right]$

32.** $\frac{2x}{x+2} - 6x\frac{1}{3x+9} = -\frac{1}{x^2 + 5x + 6}$ $\left[-\frac{1}{2}\right]$

33.** $\frac{2u-3}{4} - \frac{1-5u}{3} < 2 - 4\frac{u-u^2}{3u}$ $\left[u < \frac{21}{10}\right]$

34.** $6u + (3u+1)^2 > (3u-1)\left(3u + \frac{1}{2}\right) - 6u$ $\left[u > -\frac{1}{13}\right]$

35.** $\frac{8}{3x} + \frac{20}{2x+8} = \frac{6x}{x^2}$ $\left[2\right]$

36.** $x(15x-7) > -\left(\frac{2}{3} - 3x\right)\left(5x - \frac{3}{2}\right)$ $\left[x > \frac{6}{5}\right]$

37.** $\frac{2(3-x)}{3} - \frac{5(1-2x)}{6} < \frac{4-2x}{2} + 1$ $\left[x < \frac{11}{12}\right]$

38.** $-\frac{2\,(9-3x)}{3} < -(x+2)(x+3) + (x+2)(x-3)$ $\left[x < -\frac{3}{4}\right]$

39.** $8x^2 + 2\left(\frac{1}{5} - 2x\right)(1+2x) > 0.$ $\left[x < \frac{1}{8}\right]$

40.** $\frac{z-3}{5} - \frac{2z+1}{3} - 1 < \frac{4z-1}{15} + 2z$ $\left[z > -\frac{28}{41}\right]$

41.** $\frac{x}{5} - \frac{x-3}{4} - \frac{x-1}{2} > \frac{1}{10} - \frac{3x+1}{4}$ $\left[x > -7\right]$

42.** $1 - \frac{5x-5}{3} < \frac{2x-3}{4} + \frac{1}{2} + \frac{x-1}{3}$ $\left[x > \frac{13}{10}\right]$

43.** $\frac{1}{2}x - \frac{6x-3}{6} > \frac{x-1}{3} + 2x + 5$ $\left[x < -\frac{25}{17}\right]$

44.** $\frac{7x-3}{4} - \frac{2x-1}{3} - \frac{1}{2} < \frac{3-2x}{6} - \frac{1}{4}$ $\left[x < \frac{14}{17}\right]$

45.** $\frac{1}{6} - \frac{(4a-1)}{3} < \frac{(2a+4)}{8} - \frac{(6-5a)}{3}$ $\left[a > \frac{8}{13}\right]$

46.** $\left(k + \frac{1}{2}\right)^2 - (k-1)^2 - 6k < 5k - 3$ $\left[k > \frac{9}{32}\right]$

47.** $\frac{z-3}{6} - \frac{2z-1}{7} - 1 < \frac{2z-1}{3} - 2z$ $\left[z < \frac{43}{51}\right]$

48.** $-\frac{-6x+18}{12} + \frac{1}{2} + \frac{x-1}{3} > 1 - \frac{5x-5}{3}$ $\left[x > \frac{8}{5}\right]$

49.** $a\left(\frac{1}{3} - 3a\right) - \frac{5a-2}{2} - \frac{1}{6} > -3a^2$ $\left[a < \frac{5}{13}\right]$

50.** $\frac{2x-6}{4}-\frac{2x-1}{7}-1<\frac{1}{3}(2x-1)-2x$ $\left[x<\frac{17}{13}\right]$

51.** $10x^2-x<\left(5x+\frac{1}{5}\right)(2x-3)$ $\left[x<-\frac{3}{68}\right]$

52.** $(7t+2)\left(2t-\frac{1}{6}\right)<-t+14t^2$ $\left[t<\frac{2}{23}\right]$

53.** $6x^2>7x-2\left(\frac{1}{3}-\frac{3}{2}x\right)\left(2x-\frac{9}{6}\right)$ $\left[x<-\frac{6}{7}\right]$

II.7 Quadratic equations

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0,$$

where a, b, c are real numbers. We require $a \neq 0$ (otherwise the equation is not quadratic!) but b or c could be zero. We begin with these cases:

$$b = 0: \qquad ax^2 + c = 0 \qquad x^2 = -\frac{c}{a} \qquad \Longrightarrow \qquad x_\pm = \pm\sqrt{-\frac{c}{a}}$$

$$c = 0: \qquad ax^2 + bx = 0 \qquad x(ax + b) = 0 \qquad \Longrightarrow \qquad x = 0, \, -\frac{b}{a}$$

In some cases, a quadratic equation may be solved by *factoring,* resulting in two linear equations

$$x^2 - 8x + 15 = 0 \qquad \Longleftrightarrow \qquad (x-3)(x-5) = 0 \qquad \Longleftrightarrow \qquad x - 3 = 0 \quad \text{or} \quad x - 5 = 0$$

with solutions $x = 3, 5$.

In general, the solutions are given by the formula

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{II.7.1}$$

If the middle coefficient is even, we have the *reduced formula*:

$$b \text{ even}: \qquad ax^2 + 2kx + c = 0 \qquad x_\pm = \frac{-k \pm \sqrt{k^2 - ac}}{a}$$

The *discriminant* of equation (II.7.1) is the quantity $\Delta = b^2 - 4ac$. If Δ is positive, the equation has two distinct real solutions. The two solutions coincide if $\Delta = 0$, e.g.

$$9\,x^2 - 30\,x + 25 = (3x - 5)^2 = 0 \qquad \Delta = 30^2 - 4 \cdot 9 \cdot 25 = 900 - 900 = 0 \qquad x_\pm = \frac{5}{3}.$$

If $\Delta < 0$, equation (II.7.1), has two *complex* (or *imaginary*) solutions. In this course, you are not required to construct or manipulate complex solutions. However, you must be able to detect their presence, from the sign of the discriminant.

Quadratic equations may also be solved by *completing the square:*

$$\begin{aligned} ax^2 + bx + c = 0 \quad &\Longrightarrow \quad x^2 + 2\frac{b}{2a}x = -\frac{c}{a} \\ &\Longrightarrow \quad x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ &\Longrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2. \end{aligned}$$

Letting $y = x + b/2a$, one then solves for y an equation of the type $y^2 = A$.

A quadratic equation may be obtained by *squaring* another equation (meaning squaring both sides of it). For instance

$$x = 5 \qquad \Longrightarrow \qquad x^2 = 25.$$

The *derived equation* $x^2 = 25$ has two solutions $x = \pm 5$, but only one of them is a solution of the original equation. By contrast, squaring the equation $x - 5 = 0$ does not introduce a *different* solution, it rather creates a duplicate of the solution $x = 5$. The situation becomes more subtle with equations involving square roots. The various possibilities are exemplified in the following table

	I	*II*	*III*	IV
(i)	$2\sqrt{x} = 1 - x$	$x \leq 1$	$3 \pm 2\sqrt{2}$	one real solution: $3 - \sqrt{2}$
(ii)	$3\sqrt{x} = 1 + x$	$x \geq -1$	$(7 \pm 3\sqrt{5})/2$	two distinct real solutions
(iii)	$2\sqrt{x} = 1 + x$	$x \geq -1$	1	two identical real solutions
(iv)	$3\sqrt{x} = -1 - x$	$x \leq -1$	$(7 \pm 3\sqrt{5})/2$	no solutions
(v)	$\sqrt{x} = 1 + x$	$x \geq -1$	$(-1 \pm \sqrt{-3})/2$	no real solutions (two complex solutions).

To explain the content of the table, and to illustrate the method of solution, let us consider the equation

$$x + 2\sqrt{x} = 1.$$

- *Step 1: isolate the radical.* We rewrite the equation in such a way that the term containing the square root is isolated on the left-hand side, with a positive coefficient. This is essential, lest you will not get rid of the square root. Our equation becomes

$$2\sqrt{x} = 1 - x$$

  which is the example (i) in the table.

- *Step 2: solve the associated inequality.* Square roots are non-negative, and so the left-hand side of the equation satisfies the inequality $2\sqrt{x} \geq 0$. But then also the right-hand side must be non-negative, resulting in the inequality

$$1 - x \geq 0 \qquad \text{with solution} \qquad x \leq 1$$

  which is displayed in column II.

- *Step 3: solve the derived equation.* Now we can square both sides of the equation, and obtain a derived quadratic equation whose solutions are listed in column III. In our case, equation (i) becomes $4x = (1-x)^2$, or $x^2 - 6x + 1 = 0$, with the two real solutions $x_{\pm} = 3 \pm 2\sqrt{2}$.

- *Step 4: select the solutions.* The solutions of the derived equation — if they are real — must satisfy the inequality II. So it is possible that one or both solutions of the derived equation are to be discarded, as detailed in column IV. Using the technique described in section I.6 (or by inspection), one can verify that

$$3 + 2\sqrt{2} > 1 \qquad\qquad 3 - 2\sqrt{2} < 1$$

  and therefore equation (i) has the single real solution $x = 3 - 2\sqrt{2}$.

Note that the equation $2\sqrt{x} = x - 1$, which differs from (i) by the sign of the right-hand side, leads to the same derived quadratic equation. In other words, by squaring we *forget* the sign of the right-hand side.

However, the associated inequality is now $x \geq 1$ (which *remembers* the sign of the right-hand side), leading to a different solution: $x = 3 + 2\sqrt{2}$.

In case (iii), the derived quadratic equation $x^2 - 2x + 1 = (x-1)^2 = 0$, has zero discriminant, giving the two identical solutions $x_\pm = 1 \pm \sqrt{0}$. In case (v), the derived equation has negative discriminant, and there are no real solutions (the solutions are *complex,* in which case the inequality II does not apply).

Note the important distinction between an equation with *no solutions* (case (iv): the derived equation has two real solutions, none of which is a solution of the original equation) and an equation with *no real solutions* (case (v): the derived equation has two complex solutions, both of which are solutions of the original equation).

Squaring an equation may introduce spurious solutions

Solve.

1. $-x^2 = -64; \qquad y^2 - 32 = 0$ $\qquad \left[\pm 8; \quad \pm 4\sqrt{2}\right]$

2. $2x^2 = 42; \qquad 25 = 9k^2$ $\qquad \left[\pm\sqrt{21}; \quad \pm\frac{5}{3}\right]$

3. $7 = \dfrac{91}{s^2}; \qquad \left(\dfrac{1}{p}\right)^2 = 225$ $\qquad \left[\pm\sqrt{13}; \quad \pm\frac{\sqrt{15}}{15}\right]$

4. $\dfrac{a-3}{a-4} = \dfrac{a-1}{a-4}$ $\qquad$ [no solution]

5. $24x^2 + 48x = 0; \qquad -56a = -32a^2$ $\qquad \left[-2, 0; \quad 0, \frac{7}{4}\right]$

6. $35b^2 - 63b = 0; \qquad 0 = -x - 12x^2$ $\qquad \left[0, \frac{9}{5}; \quad -\frac{1}{12}, 0\right]$

Solve for the variable in curly brackets.

7. $x^2 - a^2 = 0 \qquad \{x\}$ $\qquad \left[\pm a\right]$

8. $y^2 - 32 = 0 \qquad \{y\}$ $\qquad \left[\pm 4\sqrt{2}\right]$

9. $2a = \dfrac{25b^4}{a} \quad (b \neq 0) \qquad \{a\}$ $\qquad \left[\pm\frac{5b^2\sqrt{2}}{2}\right]$

10. $z^2 + zu^2 = uz + u^3 \qquad \{z\}$ $\qquad \left[u, -u^2\right]$

11. $ax + b + \dfrac{c}{x} = 0 \quad (ac \neq 0) \qquad \{x\}$ $\qquad \left[\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

12.** $az^2 - 13az - bc^2z = -13bc^2$ $\{z\}$ $\left[13, \frac{bc^2}{a}\right]$

13.** $d^2 - da - abcd = -a^2bc$ $\{d\}$ $\left[abc, a\right]$

Solve by factoring.

14. $x^2 - 12x + 27;$ $y^2 - 4x - 45 = 0$ $\left[3, 9; \quad -5, 9\right]$

15. $a^2 = 4a - 4;$ $-b^2 = 10b + 21$ $\left[2; \quad -7, -3\right]$

16. $4z^2 + 15 = -17z;$ $14 - 11w = 9w^2$ $\left[-3, -\frac{5}{4}; \quad -2, \frac{7}{9}\right]$

17. $121 + 44w = -4w^2;$ $64z^2 - 112z + 49 = 0$ $\left[-\frac{11}{2}; \quad \frac{7}{8}\right]$

Solve by completing the square.

18. $x^2 - 4x + 3 = 0;$ $x^2 + 6x + 5 = 0$ $\left[1, 3; \quad -5, -1\right]$

19. $12z^2 + 24z = 36;$ $52 + 26k = 13k^2$ $\left[-3, 1; \quad 1 \pm \sqrt{5}\right]$

20. $x^2 + 6x - 8 = 0;$ $r^2 + 24r = -104$ $\left[-3 \pm \sqrt{17}; \quad -12 \pm 2\sqrt{10}\right]$

21. $z^2 - 14z + 1 = 0;$ $a^2 - 20a + 12 = 0$ $\left[7 \pm 4\sqrt{3}; \quad 10 \pm 2\sqrt{22}\right]$

Solve.

22. $\frac{3}{r-4} - 5\frac{1}{r+4} = 1$ $\left[-8, 6\right]$

23. $\frac{x-1}{x-2} = \frac{x-3}{x-4}$ $\left[\text{no solution}\right]$

24. $\frac{1}{2+3s} = \frac{s}{2+s}$ $\left[-1, \frac{2}{3}\right]$

25. $1 + \frac{12}{x^2-16} = \frac{5}{x-4}$ $\left[-3, 8\right]$

26. $25\,\frac{1}{y-3} = -3 + y$ $\left[-2, 8\right]$

27. $z = (z+1)^2$ $\left[\text{no real solution}\right]$

28. $x^2 + 2x - 1 = 0$ $\left[-1 \pm \sqrt{2}\right]$

29. $a^2 - 16a = 8$ $\left[8 \pm 6\sqrt{2}\right]$

30. $3x^2 = 2x + 3$ $\left[\dfrac{1 \pm \sqrt{10}}{3}\right]$

31. $3y^2 = 9y + 7$ $\left[\dfrac{3}{2} \pm \dfrac{\sqrt{165}}{6}\right]$

32.* $20 = 3a^2 - 32a$ $\left[\dfrac{2}{3}\left(8 \pm 2\sqrt{79}\right)\right]$

33.* $u^2 = 2u + 566$ $\left[1 \pm 9\sqrt{7}\right]$

34.** $3\dfrac{x-1}{6x-12} = \dfrac{-21-7x}{14x}$ $\left[\pm\sqrt{3}\right]$

35.** $1 - \dfrac{1}{2y}\left(-1 + y^2\right) = \dfrac{y}{5}$ $\left[\dfrac{5 \pm 2\sqrt{15}}{7}\right]$

36.** $\dfrac{5}{x} - \dfrac{3}{1-2x^2} = \dfrac{1}{2x}$ $\left[\dfrac{-1 \pm \sqrt{19}}{6}\right]$

37.** $\dfrac{2}{x^2} - \dfrac{1}{6x-1} = \dfrac{1}{x^2}$ $\left[3 \pm 2\sqrt{2}\right]$

38.** $X - \dfrac{(3X-1)^2}{2X} = -(4X+6)\left(\dfrac{3}{2X} - \dfrac{1}{X}\right)$ $\left[\dfrac{5 \pm 2\sqrt{15}}{7}\right]$

39.** $4x - \dfrac{x^2-9}{x+3} = \dfrac{6}{3x-1}$ $\left[\dfrac{-1 \pm \sqrt{10}}{3}\right]$

40.** $\dfrac{1-x^2}{x+1} = \dfrac{5}{10x-35}$ $\left[\dfrac{9 \pm \sqrt{17}}{4}\right]$

41.** $\dfrac{1}{3} - \dfrac{4x-1}{16x^2-1} = \dfrac{2x-1}{x}$ $\left[-\dfrac{3}{10}, \dfrac{1}{2}\right]$

42.** $3\dfrac{5}{15\,x^2} + \dfrac{x+1}{x^2+x} - \dfrac{6}{18} = \dfrac{2}{x^2} - \dfrac{3}{12}$ $\left[6 \pm 2\sqrt{6}\right]$

43.** $\dfrac{1}{x} + \dfrac{5x-1}{x-2x} = \dfrac{x+3}{3}$ $\left[-9 \pm \sqrt{87}\right]$

Determine all real solutions.

44. $\sqrt{x} + 1 = 0;\quad x + \sqrt{x+2} = 0$ $\left[\text{no solution};\quad -1\right]$

45. $\sqrt{z+8} = z - 4;\quad 2\sqrt{y+4} = y + 1$ $\left[8;\quad 5\right]$

46. $2 + c = \sqrt{6c+7};\quad 1 + 2\sqrt{5x-6} = 2x + 1$ $\left[-1,3;\quad 2,3\right]$

47. $\sqrt{35-5x} = x - 7;\quad \sqrt{t^2+7} - t + 5 = 0$ $\left[7;\quad \text{no solution}\right]$

48. $\sqrt{(2x+1)(x-1)} = x-1; \qquad 3x-\sqrt{x} = 2(x-1)$ $\left[1; \quad \text{complex solutions}\right]$

49.** $x+3\sqrt{x-2} = 0$ $\left[\text{no solution}\right]$

50.** $0 = \dfrac{1}{\sqrt{5-2t}} - \dfrac{1}{1-2t}$ $\left[\dfrac{1-\sqrt{17}}{4}\right]$

51.** $2\sqrt{x+1}+x = 0$ $\left[2-2\sqrt{2}\right]$

52.** $2x-1-\sqrt{3x+3} = 0$ $\left[2\right]$

53.** $4\sqrt{x-2} = 1-x$ $\left[\text{no solution}\right]$

54.** $\sqrt{2x-1} = \dfrac{x}{6}$ $\left[6\left(6\pm\sqrt{35}\right)\right]$

55.** $\dfrac{2-3x}{\sqrt{x}} = 1$ $\left[\dfrac{4}{9}\right]$

56.** $\dfrac{4}{2x-2} = \dfrac{x}{2}\left(\dfrac{5}{3x^2}+\dfrac{1}{x}\right)$ $\left[\dfrac{5\pm 2\sqrt{10}}{3}\right]$

57.** $-\dfrac{1}{\sqrt{3x+5}} + \dfrac{1}{x} = 0$ $\left[\dfrac{3+\sqrt{29}}{2}\right]$

58.** $\dfrac{3x^2+3x}{x+1} - \sqrt{x} = 2x\sqrt{2}$ $\left[0, (3+2\sqrt{2})^2\right]$

59.** $\dfrac{1-x}{3\sqrt{x-2}} = 1$ $\left[\text{no solution}\right]$

60.** $\dfrac{4}{x}\left(-1+\sqrt{10x-3}\right) = 5$ $\left[\dfrac{4}{5}\left(3\pm\sqrt{5}\right)\right]$

61.** $\dfrac{5a-55}{5} = 3\dfrac{a-9}{\sqrt{9a-81}}$ $\left[13\right]$

62.** $\dfrac{1}{x} + \dfrac{1}{\sqrt{6x-1}} = 0$ $\left[\text{no solution}\right]$

63.** $\dfrac{-x}{\sqrt{2x+5}} = 1$ $\left[1-\sqrt{6}\right]$

64.** $\dfrac{2}{x} + \dfrac{1}{\sqrt{x+1}} = 0$ $\left[2-2\sqrt{2}\right]$

65.** $\dfrac{1}{3x} = \dfrac{1}{\sqrt{9x+9}}$ $\left[\dfrac{1+\sqrt{5}}{2}\right]$

66.** $0 = \frac{1}{\sqrt{5-2w}} - \frac{3}{1-3w}$ $\left[-\frac{2}{3}(1+2\sqrt{3})\right]$

67.** $\frac{2}{\sqrt{4x^2-24x+36}} = \frac{x}{x^2-3x}$ $\left[\text{indeterminate}\right]$