Rigidity of Frameworks

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A graph $G$ is pair $(V, E)$ where $V$ and $E$ are sets and the elements of $E$ are unordered pairs of elements of $V$. We call the elements of $V$ *vertices* and elements of $E$ *edges.*
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A (2-dimensional) framework is pair $(G, p)$ where $G$ is a graph, and $p : V \rightarrow \mathbb{R}^2$. We think of the edges as ‘metal bars’ and each vertex as a ‘universal joint’ which allows the bars incident to it to rotate in any direction.
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A framework $(G, p)$ is generic if the vertices are at ‘generic points’ in the plane i.e. the coordinates of the points $p(v)$, $v \in V$, are algebraically independent over $\mathbb{Q}$. Intuitively this means that there are no ‘special relationships’ between the points e.g. no three points lie on a line.
Rigidity

The framework \((G, p)\) is **rigid** if every continuous motion of the vertices which preserves the lengths of the edges, must preserve the distances between ALL pairs of vertices.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4
Consider a motion of a framework \((G, p)\). At time \(t = 0\), each vertex \(v\) will have a velocity \(q(v)\).
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This gives us a system of \(|E|\) linear equations for the \(2|V|\) coordinates of the velocities. Any \(q : V \to \mathbb{R}^2\) which satisfies this system of equations is an **infinitesimal motion** of \((G, p)\).
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This gives us a system of \(|E|\) linear equations for the \(2|V|\) coordinates of the velocities. Any \(q : V \rightarrow \mathbb{R}^2\) which satisfies this system of equations is an \textit{infinitesimal motion} of \((G, p)\).

The \(|E| \times 2|V|\) matrix of coefficients of this system of equations is the \textit{rigidity matrix} \(R(G, p)\) for \((G, p)\).
Example

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$$\begin{pmatrix}
  p(v_1) - p(v_2) & p(v_2) - p(v_1) & 0 & 0 \\
  0 & p(v_2) - p(v_3) & p(v_3) - p(v_2) & 0 \\
  0 & 0 & p(v_3) - p(v_4) & p(v_4) - p(v_3) \\
  p(v_1) - p(v_4) & 0 & 0 & p(v_4) - p(v_1) \\
  0 & p(v_2) - p(v_4) & 0 & p(v_4) - p(v_2)
\end{pmatrix}$$
The null space $Z(G, p)$ of $R(G, p)$ is the set of all infinitesimal motions of $(G, p)$. 
A rank condition for rigidity

- The null space $Z(G, p)$ of $R(G, p)$ is the set of all infinitesimal motions of $(G, p)$.

- We always have $\dim Z(G, p) \geq 3$ since every framework has three linearly independent infinitesimal motions (e.g. translations along the two axes and a rotation about the origin). Hence $\text{rank } R(G, p) \leq 2n - 3$ where $n = |V|$. 
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Asimow and Roth showed that this sufficient condition for rigidity is also necessary when $(G, p)$ is generic.

**Theorem (Asimow and Roth, 1979)**

If $(G, p)$ is generic, then $(G, p)$ is rigid if and only if $\text{rank } R(G, p) = 2n - 3$. 
A set of edges $F$ in a framework $(G, p)$ is \textit{independent} if the rows of $R(G, p)$ corresponding to $F$ are linearly independent.
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Characterisation of generic rigidity

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- We can calculate $\text{rank } R(G, p)$ if we can determine when a given set of edges of $G$ is independent.
- We know that $\text{rank } (G, p) \leq 2|V| - 3$. This implies that if $F$ is independent then, for all $X \subseteq V$ with $|X| \geq 2$, the number of edges of $F$ which join the vertices in $X$ is at most $2|X| - 3$. 

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G. Laman showed that this necessary condition for independence is also sufficient when $(G, p)$ is generic.

**Theorem (Laman 1970)**

A set of edges $F$ in a generic framework $(G, p)$ is independent if and only if for all $X \subseteq V$, the number of edges of $F$ which join the vertices in $X$ is at most $2|X| - 3$. 
Algorithm for constructing a maximum independent set of edges

Given a framework \((G, p)\) we can ‘greedily’ grow a largest independent set of edges \(F\) as follows.
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- **INITIAL STEP** Choose an arbitrary edge and put it in \(F\).
- **RECURSIVE STEP** Choose an edge \(e\) which has not yet been considered and use Laman’s theorem to check whether \(F + e\) is independent:
  - If it is, put \(e\) in \(F\);
  - If it isn’t, delete \(e\) and move on to another edge.

Stop when all edges have been considered and output \(F\).
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**NOTE.** \((G, p)\) is rigid if and only if \(|F| = 2n - 3\).