Beyond complex numbers

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We have gradually expanded the set of numbers we use: first from finger counting to the whole set of positive integers, then to positive rationals, irrational reals, negatives and finally to complex numbers. It has not always been easy to accept new numbers. Negative numbers were rejected for centuries, and complex numbers, the square roots of negative numbers, were considered impossible.

Complex numbers behave like ordinary numbers. You can add, subtract, multiply and divide them, and on top of that, do things which you cannot do with real numbers. Today, complex numbers have many important applications in mathematics and physics, and scientists could not live without them.

What if we take the next step? What comes after the complex numbers? Is there a bigger number system that has the same nice properties as the real numbers and the complex numbers?

The answer is yes. In fact, there are two (and only two) bigger number systems that resemble real and complex numbers, and their discovery has been almost as dramatic as that of the complex numbers.

1 Complex numbers

Complex numbers where discovered in the 16th century when Italian mathematicians tried to find a general solution to the cubic equation

$$x^3 + ax^2 + bx + c = 0.$$

At that time, mathematicians did not publish their results but kept them secret. They made their living by challenging each other to public contests of problem solving in which the winner got money and fame. In these contests it was useful to know things other people did not know. Also the formula for solving cubic equations, or at least a part of it, was first kept secret.

The mathematicians of the time did not like negative numbers because to them they had no meaning. What does minus three potatoes mean? The square root of a negative number was even worse. It was impossible. These numbers were referred as fictitious, absurd or false.

However, mathematicians noticed that sometimes when they used the formula for the cubic equation, they had square roots of negative numbers in the intermediary steps. When they were brave enough to treat these as ordinary numbers for which the familiar laws of arithmetic hold, they noticed that despite the impossible intermediary steps, the resulting root was in some cases a real number.

Even though mathematician started to use the square roots of negative numbers, they did not know what their geometrical interpretation was. It was not before the 19th century before mathematicians realised how to treat them as points of the so-called complex plane.

Definition 1. *The set of complex numbers consists of elements of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.*

The rules of arithmetic are defined as follows:

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Subtraction: $(a + bi) - (c + di) = (a - c) + (b - c)i$

Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

Division:

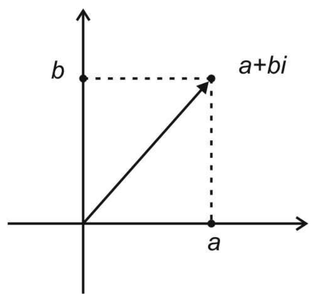
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{-ad + bc}{c^2 + d^2} i.$$

The Irish mathematician William Hamilton noticed that the complex number $a + bi$ can be written as a pair of real numbers (a, b) – or, a *vector*. For these vectors it is possible to define the scalar multiplication by real numbers, that is,

$$r \cdot (a, b) = (ra, rb)$$

for all $r, a, b \in \mathbb{R}$. This means that the set of complex numbers is just the real plane $\mathbb{R}^2$ with multiplication defined in quite a weird way:

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$



For each complex number $a+bi$, one can define its *norm* or *length* $|a+bi|$, which is just the length of the corresponding vector. In other words, we have $|a + bi| = \sqrt{a^2 + b^2}$. The norm is compatible with multiplication:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|.$$

The *complex conjugate* of the complex number $a + bi$ is $\overline{a + bi} = a - bi$. The rule for dividing complex numbers can now be expressed using lengths and complex conjugates:

$$\frac{z_1}{z_2} = \frac{1}{|z_2|^2} \cdot z_1 \cdot \overline{z_2},$$

where $z_1, z_2 \in \mathbb{C}$.

All these properties make complex numbers a very special number system, a *normed division algebra.* Complex numbers have a *norm* (or length), and one can *divide* complex numbers. We will now start to look for other normed division algebras, that is, other number systems that have similar properties as complex numbers.

2 Normed division algebras

This section is for those who want something more precise, and it can be skipped.

A normed division algebra over the real numbers is the space $\mathbb{R}^m$ with multiplication and division. (The space $\mathbb{R}^m$ consists of finite sequences of

m real numbers $x = (x_1, x_2, \ldots, x_m)$.) The addition, multiplication by real numbers and the norm (or length) of a vector are defined as usual, and all these operations behave in a nice way. In other words, in a normed division algebra the following rules hold:

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$
$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$(ax) \cdot (by) = (ab) \cdot (xy),$$

where x, y and z are elements of $\mathbb{R}^m$ and a and b are real numbers.

For every element u and any non-zero element v there exists precisely one element x such that $u = v \cdot x$ and precisely one element y such that $u = y \cdot v$. This means that we can divide elements of a normed division algebra.

Each element x of a normed division algebra has norm (or length) $|x|$, which is a real number that satisfies the rule

$$|xy| = |x||y|.$$

Real and complex numbers are normed division algebras, and we will now define two other division algebras. The aim is to find all such structures.

3 Higher dimensions

Now we have two nice number systems, $\mathbb{R}$ and $\mathbb{C}$, in which it is possible to add, subtract, multiply and divide. Also, in both cases we can define the norm of an element. (In the case of real numbers the norm is just the absolute value.)

The complex numbers are a 2-dimensional structure because every complex number can be written as a sequence of two real numbers. The set of real numbers is of course a 1-dimensional structure because every real number can be written as a sequence of one real number.

Now that we have a 1-dimensional and a 2-dimensional number system, it is natural to ask whether there is a 3-dimensional number system that has the same nice properties as the real numbers and complex numbers?

In a case of an n-dimensional structure, it is easy to find out how to add two things together, define the norm or multiply elements with real numbers. It can be done in exactly the same way as in the case of complex numbers.

The only difficult thing is the multiplication. You need to define it in such a way that you can also divide. For example, one might think that the obvious way of defining the multiplication of pairs would be $(a, b) \cdot (c, d) = (a \cdot c, b \cdot d)$. However, in this case one cannot divide by pairs that have zero in one component. For instance, we would have

$$\frac{(2,1)}{(1,0)} = \left(\frac{2}{1}, \frac{1}{0}\right),$$

which is impossible. This explains why the multiplication of complex numbers is defined in such a complicated way.

Hamilton had realised how to treat complex numbers as pairs of real numbers, and after that he wanted to find triples that would behave in a similar manner than complex numbers.

Hamilton tried very hard for eight years, but could not define the multiplication for his triples. In 1843 he finally realised that he actually needed four dimensions. Hamilton was so thrilled about his discovery that he carved the rules of multiplication of his 4-dimensional structure into a stone of Brougham Bridge in Dublin.

4 Quaternions $\mathbb{H}$

The structure Hamilton came up with is called quaternions (or Hamiltonians). One takes *three* square roots of -1: i, j and k. All the possible combinations of 1, i, j, and k form the set of quaternions.

Definition 2. *Quaternions are of the form $a + bi + cj + dk$, where a, b, c, d are real numbers and $i^2 = j^2 = k^2 = -1$.*

Each quaternion $a + bi + cj + dk$ can be written as a list of four real numbers (a, b, c, d). This means that the quaternions are a 4-dimensional structure.

As mentioned above, addition, multiplication by real numbers and norm are easy to define. They are defined in the same way as in the case of complex numbers:

a) $(a+bi+cj+dk)+(a'+b'i+c'j+d'k) = (a+a')+(b+b')i+(c+c')j+(d+d')k$

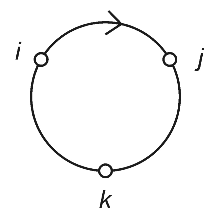
b) $r \cdot (a + bi + cj + dk) = ra + (rb)i + (rc)j + (rd)k$, where $r \in \mathbb{R}$

c) $|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

The multiplication table is defined by the famous equations of Hamilton:

$$\begin{aligned}
&i^2 = j^2 = k^2 = -1,\\
&ij = k, jk = i, ki = j,\\
&ji = -k, jk = -i, ik = -j.
\end{aligned}$$

The rules of multiplication can be visualised with the following picture in which the arrow indicates the sign of the product.



Using these equations, one can calculate the product of two arbitrary quaternions. For example, we have

$$(2 + 3i) \cdot (3i - j) = 6i - 2j - 9 - 3k$$

and

$$\begin{aligned}
&(1 - 4j + 3k) \cdot (-2i + j + 2k)\\
&= -2i + j + 2k - 8k + 4 - 8i - 6j - 3i - 6\\
&= -2 - 13i - 5j - 6k.
\end{aligned}$$

One can also divide quaternions by using norms and conjugates just as in the case of complex numbers. The conjugate of the quaternion $a+bi+cj+dk$ is $\overline{a + bi + cj + dk} = a - bi - cj - dk$. Now we have

$$\frac{x}{y} = \frac{1}{|x|^2} \cdot x \cdot \overline{y}$$

for all quaternions x and y.

Notice that unlike the multiplication of real and complex numbers, the multiplication of quaternions is *not commutative*. In other words, in a product you cannot change the order of the factors. For example, we have $ij = k$ but $ji = -k$.

Hamilton was obsessed by quaternions, and devoted the rest of his life to them. For a while the quaternions were very popular, and much of what is now done with three-dimensional vectors was then done with quaternions. Not everybody thought it was a good thing, and there was a fierce battle between those who wanted to use vectors and those who wanted to use quaternions. For example, the great physicists Kelvin and Heaviside wrote some devastating attacks against quaternions. It was the quaternions that finally lost the battle.

5 Even higher dimensions

Hamilton's friend John Graves was interested in quaternions, and thought that there might be bigger structures with same kind of miraculous properties. Indeed, only a couple of months after Hamiltons great discovery, Graves constructed an interesting 8-dimensional number system. Graves sent letters describing his results to Hamilton who was impressed, and promised to publish Graves' discovery.

However, Hamilton was very busy, and did not publish the results right away. In 1845, young Englishman Arthur Cayley came up with similar ideas, and published a paper in which he gave a description of the same 8-dimensional number system Graves had discovered earlied. Graves was of course upset, and pointed out that he had known about the structure since 1843. But it was too late, and even though it was admitted that Graves had invented the number system first, people had already started to call it Cayley numbers.

6 Octonions $\mathbb{O}$

Nowadays the 8-dimensional number system found by Graves and Cayley is called *octonions*. This time one takes *seven* square roots of -1: $i_0, i_1, \dots i_6$. All the possible combinations of 1 and these roots form the set of octonions.

Definition 3. *Octonions are of the form*

$$a + a_0 i_0 + \cdots + a_6 i_6,$$

where $a, a_0, \dots, a_6$ are real numbers and $i_t^2 = -1$ for all t.

Each octonion $a + a_0 i_0 + \cdots + a_6 i_6$ can be written as a list of eight real numbers $(a, a_0, \ldots a_6)$. This means that the octonions are an 8-dimensional structure.

Again, addition, multiplication by real numbers and norm are defined in the familiar way:

addition:

$$\begin{aligned}&(a + a_0 i_0 + \cdots + a_6 i_6) + (a' + a_0 i_0' + \cdots + a_6' i_6)\\ &= (a + a') + (a_0 + a_0')i_0 + \cdots + (a_6 + a_6')i_6\end{aligned}$$

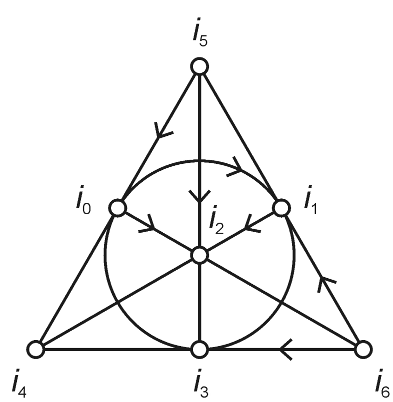
multiplication by real numbers:

$$r \cdot (a + a_0 i_0 + \cdots + a_6 i_6) = ra + (ra_0)i_0 + \cdots + (ra_6)i_6,\ r \in \mathbb{R}$$

norm:

$$|a + a_0 i_0 + \cdots + a_6 i_6| = \sqrt{a^2 + a_0^2 + \cdots + a_6^2}$$

To define the multiplication table, it is enough to do it for the octonions $i_0, i_1, \ldots i_6$. The rules of multiplication are described in the picture below. The product of two octonions is the third octonion that is on the same line. The arrow indicates what the sign of the product is. For example, the octonions i_1, i_2 and i_4 are on the same line, so $i_1 i_2 = i_4$, $i_2 i_1 = -i_4$, $i_1 i_4 = -i_2$ and so on.



Notice that every line in the diagram behaves like the quaternions i, j and k.

There is also another way of expressing the rule of multiplication:

the octonions i_t, i_{t+1}, i_{t+3} behave as the quaternions i, j, k.

The subscripts are understood modulo 7. For example, consider the product $i_5 \cdot i_6$. Here t equals 5, so $t + 3$ equals 8. Because $8 - 7 = 1$, the product is equal to i_1. The triples (i_t, i_{t+1}, i_{t+3}) correspond to the lines of the above diagram.

Octonions can be divided by using norms and conjugates in the same way as in the case of complex numbers and quaternions.

Notice that unlike multiplication of real numbers, complex numbers and quaternions, the multiplication of octonions is *not associative.* In other words, the place of the brackets in the octonion multiplication matters. For example, we have $(i_0 i_1) i_2 = i_3 i_2 = -i_5$ but $i_0(i_1 i_2) = i_0 i_4 = i_5$.

7 Other dimensions

Now we have four structures of four different dimensionalities:

1) Real numbers – dimension 1

2) Complex numbers – dimension 2

3) Quaternions – dimension 4

4) Octonions – dimension 8

What about other dimensions? Graves considered the idea of a general theory of 2^m-ions, but failed to construct even a 16-dimensional number system in which it would be possible to divide. In other words, he could not construct any other normed division algebras. Graves started to think that it might be impossible, and he was right.

Every number system in our list can in some sense be built from the real numbers. For example $\mathbb{C} = \mathbb{R} + \mathbb{R}i$. Therefore the dimension of $\mathbb{C}$ is the dimension of $\mathbb{R}$ plus the dimension of $\mathbb{R}$, that is, twice the dimension of $\mathbb{R}$. The quaternions, on the other hand, are built from complex numbers ($\mathbb{H} = \mathbb{C} + \mathbb{C}j$). The dimension of quaternions is twice the dimension of $\mathbb{C}$. Finally, the octonions are built from quaternions, and therefore the dimension of octonions is twice the dimension of the quaternions. This way we can obtain new number systems by doubling the old ones.

What happens after octonions? If we double octonions, we obtain a 16-dimensional set, but in this new set it is not possible to divide. Each number system in our list is contained in the next one. When passing from a number system to the next one, we always lose something, and this is why the doubling process cannot be continued after octonions. In some sense, octonions are too weird for doubling.

What kind of things do we lose? First we lose trivial conjugation. For real numbers the complex conjugation $a + bi \mapsto a - bi$ does not do anything, but for complex numbers it does. In other words, for the complex numbers complex conjugation is non-trivial. Then we lose commutativity because in quaternion multiplication the order of the factors matters. Finally, we lose associativity because in octonion multiplication the place of the brackets matters. It can be shown that it is the non-associativity of octonions that makes dividing in the 16-dimensional set impossible. Therefore it is possible to divide only in dimensions 1, 2, 4 and 8.

8 Literature

- John Baez: The octonions
  http://math.ucr.edu/home/baez/octonions/

- Helen Joyce: Curious quaternions
  http://plus.maths.org/issue32/features/baez/

- Helen Joyce: Ubiquitous octonions
  http://plus.maths.org/issue33/features/baez/index.html

- Paul Nahin: An imaginary tale: The story of $\sqrt{-1}$