Hypercomplex numbers

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The family of normed division algebras

- Real numbers
- Complex numbers
- Quaternions
- Octonions
History of complex numbers

- Italy in the 15th century
- Discovered when looking for a general solution to cubic equations
  \[ x^3 + ax^2 + bx + c = 0 \]
- Geometrical meaning in the 18th century
Complex numbers

- $a + bi$, where $i = \sqrt{-1}$
- Addition: $(a + bi) + (a' + b'i) = (a + a') + (b + b')i$
- Multiplication by reals: $r(a + bi) = (ra) + (rb)i$
- Multiplication: $(a + bi) \cdot (a' + b'i) = (aa' - bb') + (ba' + ab')i$
Complex numbers

- Can be written as pairs of real numbers: \((a, b)\)
- Two dimensional space \(\mathbb{R}^2\)
  - Addition: \((a, b) + (a', b') = (a + a', b + b')\)
  - Multiplication by reals: \(r(a, b) = (ra, rb)\)
  - Multiplication: \((a, b) \cdot (a', b') = (aa' - bb', ba' + ab')\)
- You can multiply vectors!
Complex numbers

- Norm (or length): \( N(a + bi) = a^2 + b^2 \)
- Complex conjugate: \( \overline{a + bi} = a - bi \)
- Division: \( (a + bi)/(a' + b'i) = (a + bi)(a' + b'i)/N(a' + b'i) \)
- You can also divide!
Normed division algebras

- $\mathbb{R}^n$ with multiplication
  - consists of lists of $n$ real numbers $(x_1, x_2, \ldots, x_n)$
- Addition and norm defined in a familiar way
- Multiplication must be such that you can also divide
Higher dimensions

- Real numbers are a 1-dimensional normed division algebra
- Complex numbers are a 2-dimensional normed division algebra
- Is there a 3-dimensional normed division algebra?
- Hamilton in 1843 in Dublin
Higher dimensions

- There is no 3-dimensional normed division algebra because you cannot define multiplication and division
Quaternions $\mathbb{H}$

- Three square roots of $-1$: $i, j, k$
- $a + bi + cj + dk$, where $a, b, c, d$ are real numbers
- 4-dimensional: $(a, b, c, d)$
Quaternions $\mathbb{H}$

- **Addition:**
  \[
  (a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k
  \]

- **Multiplication by reals:**
  \[
  r \cdot (a + bi + cj + dk) = (ra) + (rb)i + (rc)j + (rd)k
  \]

- **Norm:**
  \[
  N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2
  \]

- **Conjugation:**
  \[
  \overline{a + bi + cj + dk} = a - bi - cj - dk
  \]
Quaternion multiplication

- Multiplication: $ij = k, \; jk = i, \; ki = j, \; ji = -k, \; jk = -i, \; ik = -j$
- Division: $x/y = x\bar{y}/N(y)$
Quaternion multiplication
Quatertion multiplication

- Quaternion multiplication is not commutative: $ij \neq ji$
What do we have this far?

- Real numbers – 1-dimensional
- Complex numbers – 2-dimensional
- Quaternions – 4-dimensional
Higher dimensions

- Is there a 8-dimensional normed division algebra?
- Hamilton’s friend Graves in 1843
Octonions

- Seven square roots of $-1$: $i_0, i_1, \ldots, i_6$
- $a + a_0i_0 + \cdots + a_6i_6$
  where $a, a_0, \ldots, a_6$ are real numbers
- 8-dimensional: $(a, a_0, \ldots, a_6)$
Octonions \( \mathbb{O} \)

- Addition, norm and conjugation as before
- Multiplication: \( i_t, i_{t+1} \) and \( i_{t+3} \) behave as the quaternions \( i, j, k \)
Octonion multiplication
Octonion multiplication

- Octonion multiplication is not associative
Higher dimensions

• Real numbers – 1-dimensional
• Complex numbers – 2-dimensional
• Quaternions – 4-dimensional
• Octonions – 8-dimensional
• Is there a 16-dimensional normed division algebra?
Hurwitz’s theorem (1898)

- The only normed division algebras are real numbers, complex numbers, quaternions and octonions
- You can divide only in dimensions 1, 2, 4 and 8
Normed division algebras

- New normed division algebras can be built from the old ones by doubling them
  - $\mathbb{C} = \mathbb{R} + \mathbb{R}i$
  - $\mathbb{H} = \mathbb{C} + \mathbb{C}j$
  - $\mathbb{O}$ from two copies of $\mathbb{H}$
- The dimensions are doubled too!
- This is the only way of making new normed division algebras
- Why can we not double the octonions?
Normed division algebras

- We lose something at every step
  - Complex numbers have non-trivial conjugation
  - Quaternions are not commutative
  - Octonions are not associative
- A non-associative normed division algebra cannot be doubled
My research

- I am studying structures called Lie algebras, and especially the Lie algebra $E_8$
- Lie algebras are important in theoretical physics, where they can be used in describing interactions between particles
- $E_8$ is used in string theory
- $E_8$ has 248 dimensions and is quite difficult to handle
- I am looking for a new simple construction for $E_8$ that uses octonions
Further reading

- Paul Nahin: An imaginary tale: The story of $\sqrt{-1}$
- John Baez: The octonions (the beginning)
- Helen Joyce: Curious quaternions (Plus magazine)
- Helen Joyce: Ubiquitous octonions (Plus magazine)