Graph Theory. Chapter 5.

Networks.

We can think of a directed weighted graph in terms of flows. For example, we might have a connected system of pipes, and wish to consider the maximum amount of oil that can flow down each pipe. We require every edge to have a non-negative weight.

We make the assumption that a flow can only go in one fixed direction down any pipe, which is why the graph is directed. To allow flow in both directions we have to have two edges with the same ends, but opposite directions, as in a dual carriage way.

In this context we use a different vocabulary, and call the weight of an edge its *capacity*.

A *flow* f in a network associates with every edge e a real number $f(e)$ where $0 \leq f(e) \leq c(e)$. So a flow could correspond to an actual flow of oil through the pipes, the flow along each pipe being in the correct direction and within the capacity of the pipe.

Given a flow in a network, some vertices may have more oil flowing out of them than flowing in; they are called 'sources': and some may have more flowing in than flowing out; they are called 'sinks'. Formally, if v is a vertex define

$$f(v) = \sum_{e \in A_v} f(e) - \sum_{e \in B_v} f(e)$$

where A_v is the set of edges leading out of v, and B_v is the set of edges leading into v. Then v is a source for f if $f(v) > 0$, and is a sink for f if $f(v) < 0$. Note that some vertices may be neither a sink nor a source for f. Note also that $f(v)$ and $f(e)$ have been defined for v a vertex and e a edge.

Now we come to the important concept of a 'cut'. Given a flow f on a network, draw the underlying graph so that all sources are in the left hand side of the page, and all sinks are in the right hand half. Vertices that are neither sources nor sinks may be placed in either half. Draw the edges as straight lines between the vertices. Now draw a straight line down the middle of the page. This will cut every edge joining a vertex on the left to a vertex on the right. We are only interested in knowing which edges are cut, and a cutting of edges that arises in this way is called a 'cut' for f. Since we only need to know when one vertex is on the same side as another to determine the cut, a more formal definition is : a 'cut' for f is a partition of the set of vertices of the graph into two parts, with all the sinks in one part and all the sources vertices in the other. Vertices that are neither sources nor sinks can lie in either part.

Because the sources are to the left and the sinks are to the right, oil will tend to flow across the cut from left to right; but some edges may take oil in the other direction.

The flow across the cut is defined to be the total rate of flow across the cut from left to right. That is, formally,

$$\sum_{e \in E} f(e) - \sum_{e \in F} f(e)$$

where E is the set of edges that are cut in which the flow is from left to right, and F is the set of edges that are cut in which the flow is from right to left. It turns out that the flow across a cut does not depend on the cut; that is to say, it does not depend on which side of the page we place the vertices that are neither sources nor sinks. More precisely:

Lemma. *Let f be a flow on a network, and let any cut for this flow be given. Then the following are equal.*

(a) $\sum_{v\in O} f(v)$*, where O is the set of vertices that are sources for f.*

(b) $-\sum_{v\in I} f(v)$*, where I is the set of vertices that are sinks for f.*

(c) The flow across the cut.

Proof. For each edge e, write $f(e)$ at the begining of the edge and $-f(e)$ at the end.

To prove that (a)=(b), add these numbers in 2 different ways.

We can add them in pairs; for each edge match the $f(e)$ at one end with the $-f(e)$ at the other. This gives the sum as zero.

Now add the same numbers, but grouping the numbers at each vertex. For a vertex that is neither a source nor a sink the sum is zero. Taking the sources together gives (a), and taking the sinks together gives -(b). So we get (a)-(b) as the answer.

Comparing the two answers gives (a)=(b).

Now sum the numbers that appear to the left of the cut.

The numbers at the ends of edges that do not cross the cut cancel, but numbers for edges that do cross the cut are not matched, so the sum is (c).

Grouping the numbers to the left of the cut by vertices, we get (a) as before.

So (a)=(c).

Call the number given by (a) (or (b) or (c)) the 'value' of the flow.

The problem that we wish to solve can be described as follows. We have a network of one-way roads, each with a given capacity. Some towns on the network need to be evacuated as fast as possible, and others will accept evacuees. Other towns will neither accept evacuees nor be evacuated. The limiting factor is the capacity of the road network. When the evacuation is taking place we will have a flow along the network, and we want to organise matters so that the flow is as great as possible. This description of the problem assumes that the towns that need to be evacuated will keep on producing evacuees as fast as the network will absorb them, and the the receiving towns will continue to be able to receive them.

Formally the problem is as follows. We have a network with two disjoint sets of vertices X and Y. We want to find a flow of maximal value which has all its sources in X and all its sinks in Y.

We will assume that all the capacities are integers, and we will only construct flows that are integer valued.

The idea is to start with the best flow that we can easily find, and keep on improving it until no further improvement is possible. All flows considered must have their sources in X and their sinks in Y.

If we can't see anything better, we can start with the zero flow.

If we move from the flow f to the flow g, and e is an edge, we have added $g(e) - f(e)$ to the flow along e. Putting $h(e) = g(e) - f(e)$, call h an 'augmenting flow', so h associates a number (an integer with our assumptions) to each edge e. We must allow $h(e)$ to be positive or negative; so it is not in general a flow.

We will construct an augmenting flow as follows. We will find an undirected path P in the network from a vertex in X to a vertex in Y. 'Undirected' means that the path is allowed to go the wrong way along an edge. For each edge e in P we will choose a non-negative integer $s(e)$ called the 'surplus' on e, and take the augmenting flow to be defined by $h(e) = s(e)$ if P goes down e in the correct direction, and $h(e) = -s(e)$ otherwise.

The first step in the algorithm is to find a suitable path. So we have a flow f in the network, and try to construct a suitable path P. Think of a refugee trying to get from a town in X to a town in Y, given the capacities of the network, and the existing flow f. The aim is to tick the towns that the refugee might arrive at. So first tick all the vertices in X, as the refugee could start from any of these towns. Now suppose that the refugee has got to a town v, and is considering getting to a town w joined by an edge e to v. If the edge goes in the direction of v to w, and if $f(e) < c(e)$, then clearly we should tick w. Now suppose that the edge goes from w to v and that $f(e) > 0$. In this case we allow the refugee to travel (illegally) in the wrong direction along e. You might think of this as follows. There is at least one refugee coming down this edge in the 'wrong' direction. As the identity of a refugee does not matter, the refugee can in effect travel in the wrong direction down this edge by exchanging identities with someone coming the other way, and neither actually moves down the edge. Proceeding in this way, tick as many vertices as can be ticked. We will show that if a vertex in Y gets ticked the flow can be increased; but if not, the flow is already optimal.

Suppose that a vertex in Y gets ticked. Then we can find an undirected path $P = (v_0, e_0, v_1, e_1, \ldots, v_n)$ such that $v_0 \in X$, and $v_n \in Y$, and if e_i is in the direction from v_i to v_{i+1} then $h(e_i) = c(e_i) - f(e_i) > 0$, and if e_i is in the direction from v_{i+1} to v_i then $h(e_i) = f(e_i) > 0$. Now let h be the minimum of all $h(e_i)$, considering all e_i in the path P. So $h > 0$. Now take the augmenting flow $h(e)$ to be h if e is an edge in the path, and to be 0 otherwise. Now using this augmenting flow increases the value of the flow by h, and all the required conditions are clearly satisfied. That is to say, the sources are in X, the sinks are in Y, the flow along each edge is in the correct direction, and does not exceed the capacity of the edge in question. The value new flow is h more than the value of the old flow.

Now suppose that no vertex in Y gets ticked. How do we prove that the flow cannot be increased?

Suppose that we find a cut, with X the sources and Y the sinks, and suppose that for every edge e with the first end to the left and the second to the right $f(e) = c(e)$, and that for every edge with the first end to the right and the second to the left $f(e) = 0$. Then the value of the flow across this cut is $\sum f(e)$, where we sum over the edges crossing the cut from left to right. Clearly we cannot find a flow that has a higher value for this cut. But if we take any flow with these sources and sinks, the value of the flow, by the Lemma, will be the value for this cut. So no flow can be found with these sources and sinks with greater value.

Now to find a suitable cut. Take the vertices on the left to be those that are ticked, and those on the right to be those that are not ticked. Then, by the rules for ticking, the result follows.

This proves that either we can find an augmenting flow, or that the flow is maximal for the given network with its sources and sinks.

Suppose now that we have a cut in our network. Some edges will cross the cut from left to right, and some from right to left (and some may not cross it at all). Clearly the flow of oil across the cut from left to right cannot exceed the number obtained by adding the capacity of every edge crossing the cut from left to right. This number is the 'capacity' of the cut. It may be the case that it is impossible to achieve this rate of flow; there may be bottle necks elsewhere.

Now take a network, with two disjoint sets of vertices X and Y. Then we have proved that the largest value of a flow where the sources are in X and the sinks are in Y is equal to the smallest capacity of any cut in which the vertices in X are to the left and those in Y are to the right.

This is called the 'max flow min cut' theorem.

The proof can be summarised is as follows. If we take any flow with X to the left and Y to the right, its value is the rate of flow across any cut, and this rate of flow is at most the capacity of the cut. So the capacity of every cut is at least the value of every flow. In particular, the smallest possible value of a cut is not less than the maximum value of a flow.

For the converse, we have shown how to construct a flow and a corresponding cut (the partition between the ticked and unticked edges) with the property that the value of the flow is the capacity of this cut. It follows that the smallest possible value of a cut cannot be greater than the maximum value of a flow.

This proves the theorem.