Polytopes and moduli of matroids over rings

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Perspectives in Lie Theory
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This talk is on joint work with Luca Moci, a second paper in preparation following up on arXiv:1209.6571.

- Matroids
- Matroids over rings, and a few applications
- The right choice of module invariants; how the axioms interrelate
- Polytopes
- The parameter space
- Coxeter generalizations?
A matroid $M$ on the finite ground set $E$ assigns to each subset $A \subseteq E$ a rank $\text{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that:

1. $\text{rk}(\emptyset) = 0$

2. $\text{rk}(A) \leq \text{rk}(A \cup \{b\}) \leq \text{rk}(A) + 1 \quad \forall A \not
\ni b$

3. $\text{rk}(A) + \text{rk}(A \cup \{b, c\}) \leq \text{rk}(A \cup \{b\}) + \text{rk}(A \cup \{c\}) \quad \forall A \not
\ni b, c$

Guiding example: realizable matroids

Let $v_1, \ldots, v_n$ be vectors in a vector space $V$.

$$\text{rk}(A) := \dim \text{span}\{v_i : i \in A\}$$
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A realizable matroid in full
Recast with \( \text{cork}(A) = r - \text{rk}(A) \).

**Definition**

A matroid \( M \) on the finite ground set \( E \) assigns to each subset \( A \subseteq E \) a corank \( \text{cork}(A) \in \mathbb{Z}_{\geq 0} \), such that:

1. \( \text{cork}(E) = 0 \)
2. \( \text{cork}(A) \geq \text{cork}(A \cup \{b\}) \geq \text{cork}(A) + 1 \quad \forall A \not\ni b \)
3. \( \text{cork}(A) + \text{cork}(A \cup \{b, c\}) \geq \text{cork}(A \cup \{b\}) + \text{cork}(A \cup \{c\}) \quad \forall A \not\ni b, c \)
Matroids over rings generalise matroids, as well as several variants which retain more data.

Valuated matroids come from configurations over a field with valuation, and remember valuations. [Dress-Wenzel]

Arithmetic matroids come from configurations over \( \mathbb{Z} \), and remember indices of sublattices. [D’Adderio-Moci]

(Compare matroids with coefficients [Dress].)
Let $R$ be a commutative ring.

Let $v_1, \ldots, v_n$ be a configuration of vectors in an $R$-module $N$. We would like a system of axioms for the quotients $N/\langle v_i : i \in A \rangle$.

### Realizable example

<table>
<thead>
<tr>
<th>$v_1 = (-2, 1)$</th>
<th>$v_2 = (1, 1)$</th>
<th>$v_3 = (4, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\emptyset$</td>
<td>$1$</td>
</tr>
<tr>
<td>$M(A)$</td>
<td>$\mathbb{Z}^2$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
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| $A$              | $3$             | $13$            |
| $M(A)$           | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z}/8$ |

| $A$              | $2$             | $23$            |
| $M(A)$           | $\mathbb{Z}/2$  | $1$             |

| $A$              | $12$            | $123$           |
| $M(A)$           | $\mathbb{Z}/3$  |                |

Fink, Moci

Polytopes and moduli of matroids over rings 7 / 20
Let $x_1, \ldots, x_n$ be a configuration of elements in an $R$-module $N$. We would like a system of axioms for the quotients $N/\langle x_i : i \in A \rangle$.

**Main definition [F-Moci]**

A matroid over $R$ on the finite ground set $E$ assigns to each subset $A \subseteq E$ a f.g. $R$ module $M(A)$ up to $\cong$, such that

for all $A \subseteq E$ and $b, c \notin A$, there are elements

$$x = x(b, c), \quad y = y(b, c) \in M(A)$$

with

$$M(A) = M(A), \quad M(A \cup \{b\}) \cong M(A)/\langle x \rangle,$$

$$M(A \cup \{c\}) \cong M(A)/\langle y \rangle, \quad M(A \cup \{b, c\}) \cong M(A)/\langle x, y \rangle.$$

Making different choices of $x$ and $y$ allows nonrealizability.
Let \( x_1, \ldots, x_n \) be a configuration of elements in an \( R \)-module \( N \). We would like a system of axioms for the quotients \( N/\langle x_i : i \in A \rangle \).

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Making different choices of $x$ and $y$ allows nonrealizability.
Matroids are matroids over fields

**Theorem 1 (F-Moci)**

Matroids over a field $k$ are equivalent to matroids*.

*if $M(E) = \emptyset$.

A f.g. $k$-module is determined by its dimension $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $k^r$, the dimension of $k^r/\langle v_i : i \in N \rangle$ is $\text{cork}(A)$.

**Example**

$$
\begin{array}{cccccccccc}
A & 0 & 1 & 2 & 12 & 3 & 13 & 23 & 123 \\
M(A) & \mathbb{R}^2 & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} & 0 & 0 & 0
\end{array}
$$

Note: The definition of matroids over $k$ is blind to which field $k$ is. For realizability the choice matters.
Matroids are matroids over fields

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**Note:** The definition of matroids over $k$ is blind to which field $k$ is.
For realizability the choice matters.
Subtorus arrangements

Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be codimension one tori in an $r$-dim’l torus $T$. [De Concini-Procesi]

The subtori $H_i = \{x : u_i(x) = 1\}$ are dual to characters $u_i \in \text{Char}(T) \cong \mathbb{Z}^r$.

Let $M$ be the matroid over $\mathbb{Z}$ represented by the $u_i$.

$$M(A) = \mathbb{Z}^k \oplus \text{(finite)} =: M(A)^{\text{free}} \oplus M(A)^{\text{torsion}}$$

Then

$$\text{rk}(M(A)^{\text{free}}) = \text{codim} \bigcap_{i \in A} H_i = \dim \text{span}\{u_i : i \in A\}$$

$$|M(A)^{\text{torsion}}| = \# \text{ components} \bigcap_{i \in A} H_i = [\mathbb{R}\{u_i\} \cap \text{Char}(T) : \mathbb{Z}\{u_i\}]$$

The arithmetic Tutte polynomial [D’Adderio-Moci] and Tutte quasipolynomial [Brändén-Moci] are invariants of $M$. 
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Tropical linear spaces

Let \((R, \text{val})\) be a valuation ring.

Given \(v_1, \ldots, v_n \in R^r\), let \(p_A = \det(v_a : a \in A)\).

The ideal of relations among the \(p_A\) is generated by Plücker relations

\[ p_{A bc}p_{A de} - p_{A bd}p_{A ce} + p_{A be}p_{A cd} = 0. \]

A **valuated matroid** remembers the \(v_A = \text{val}(p_A)\), which satisfy

\[ \min\{v_{A bc} + v_{A de}, v_{A bd} + v_{A ce}, v_{A be} + v_{A cd}\} \text{ appears twice.} \]

The Plücker relations cut out the Grassmannian.

The valuated Plücker relations define the **tropical Dressian**, whose points correspond to tropical linear spaces.

A matroid over \(R\) contains the data of a tropical linear space.

But what other data is in there?
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A matroid over \(R\) contains the data of a tropical linear space.

But what other data is in there?
Assume \((R, \text{val})\) is a valuation ring.

**Theorem**

Any finitely presented \(R\)-module is the direct sum of copies of \(R\) and \(R/\text{val}^{-1}[a, \infty)\).

Let \(\text{length}(R) = \infty\) and \(\text{length}(R/\text{val}^{-1}[a, \infty)) = a\), and extend additively.

**Proposition**

For a f.p. \(R\)-module \(N\), define

\[ t_i(N) := \min_{x_1, \ldots, x_i \in N} \text{length}(N/\langle x_1, \ldots, x_i \rangle). \]

Then the series \((t_i(N))_{i \geq 0}\) is a complete isomorphism invariant.
The length axiomatization

**Theorem**

Let \( t_i(A) \in \text{val}(R) \cup \{\infty\} \) for each \( A \subseteq E \) and \( i \geq 0 \). There exists a matroid \( M \) over \( R \) so that \( t_i(A) = t_i(M(A)) \iff \) for all \( A \subseteq E \) and \( b, c \in E \setminus A \) and \( i \geq 0 \),

\[(Ts)\] the sequence \( (t_i(A))_{i \in \mathbb{N}} \) stabilises at zero;

\[(T0)\] \( t_i(A) - t_{i+1}(A) \geq t_{i+1}(A) - t_{i+2}(A) \);

\[(T1)\] \( t_i(A) - t_{i+1}(A) \geq t_i(AB) - t_{i+1}(AB) \geq t_{i+1}(A) - t_{i+2}(A) \);

\[(T2)\] \( t_{i+1}(A) - t_{i+1}(Ab) - t_{i+1}(Ac) + t_i(ABC) \geq \min\{t_i(AB) - t_{i+1}(AB), t_i(AC) - t_{i+1}(AC)\} \),

and equality is attained if the terms of the \( \min \) differ.

Conditions \( (T0-2) \) imply the valued matroid axiom:

\[
\min\{t_i(ABC) + t_i(Ade), t_i(ABd) + t_i(Ace), t_i(Abe) + t_i(Acd)\}
\]

is attained twice. \( \text{ (D00)} \)
The length axiomatization

**Theorem**

Let \( t_i(A) \in \text{val}(R) \cup \{ \infty \} \) for each \( A \subset E \) and \( i \geq 0 \). There exists a matroid \( M \) over \( R \) so that \( t_i(A) = t_i(M(A)) \iff \) for all \( A \subset E \) and \( b, c \in E \setminus A \) and \( i \geq 0 \),

1. **(Ts)** the sequence \( (t_i(A))_{i \in \mathbb{N}} \) stabilizes at zero;
2. **(T0)** \( t_i(A) - t_{i+1}(A) \geq t_{i+1}(A) - t_{i+2}(A) \);
3. **(T1)** \( t_i(A) - t_{i+1}(A) \geq t_i(A b) - t_{i+1}(A b) \geq t_{i+1}(A) - t_{i+2}(A) \);
4. **(T2)** \( t_{i+1}(A) - t_{i+1}(A b) - t_{i+1}(A c) + t_i(A b c) \geq \min\{t_i(A b) - t_{i+1}(A b), t_i(A c) - t_{i+1}(A c)\} \),

and equality is attained if the terms of the min differ.

Conditions (T0–2) imply the valuated matroid axiom:

\[
\min\{t_i(A b c) + t_i(A d e), t_i(A b d) + t_i(A c e), t_i(A b e) + t_i(A c d)\}
\]

is attained twice. (D00)
Generizing and zeroizing

Let \( x_1, \ldots, x_n \in N \) have matroid \( M \) over \( R \).

If we add \( x_0 = 0 \in N \) to the configuration, the new matroid \( M_0 \) has
\[
M_0(A_0) = M_0(A) = M(A).
\]

If instead we add a suitably generic \( x_* \in N \), the new matroid \( M_* \) has
\[
M_*(A) = M(A), \quad t_i(M_*(A_*)) = t_{i+1}(M(A)).
\]

By specializing \( k \) elements to zero and \( \ell \) to generic, \( 0 \leq k, \ell \leq 2 \), condition (D00) becomes (D\( k\ell \)).

Fact

(T0) is (D22). (T1) is (D12) \( \land \) (D21). (T2) is (D11).

Almost-corollary

(D00), (D01), (D10), (Ts) are another choice of axioms.
Generizing and zeroizing

Let $x_1, \ldots, x_n \in N$ have matroid $M$ over $R$.

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(T0) is (D22). (T1) is (D12) $\land$ (D21). (T2) is (D11).

Almost-corollary

(D00), (D01), (D10), (Ts) are another choice of axioms.
The (basis) polytope of a usual matroid $M$ is

$$\text{conv}\{e_A : |A| = r, \cork(A) = 0\}.$$ 

The polytope of a valuated matroid $M$ is

$$\text{conv}\{(e_A, p_A) : |A| = r\} + \mathbb{R}_{\geq 0}(0, 1)$$

where $\text{conv}$ discards points $(v, \infty)$.

**Theorem**

$P$ is a (valuated) matroid polytope if and only if

- each vertex (resp. its projection to $\mathbb{R}^n$) is a 0-1 vector; and
- each edge (resp. its projection) is in some direction $e_i - e_j$, with $i, j \in [n]$. 

Fink, Moci
Let \((R, \text{val})\) be a valuation ring with \(\text{val}(R) \subseteq \mathbb{R}\). For \(M\) a matroid over \(R\), define the polytope

\[
P(M) := \text{conv}\{(e_A, i, t_i(M(A)))\} + \mathbb{R}_{\geq 0}(0, 0, 1).
\]

**Theorem**

\(P\) is the polytope of a matroid over \(R\) if and only if

- the projection of each vertex to \(\mathbb{R}^n \times \mathbb{R}\) is in \(\{0, 1\}^n \times \mathbb{N}\);
- the projection of each edge is in some direction \(e_i - e_j\), where \(i, j \in [n] \cup \{0, n + 1\}\), taking \(e_0 = 0\);
- \(P\) contains \([0, 1]^n \times [N, \infty) \times [0, \infty)\) for some \(N\).

If \(R\) has more primes, introduce more height coordinates.
Let \((R, \text{val})\) be a valuation ring with \(\text{val}(R) \subseteq \mathbb{R}\).

For \(M\) a matroid over \(R\), define the polytope

\[ P(M) := \text{conv}\{(e_A, i, t_i(M(A))\} + \mathbb{R}_{\geq 0}(0, 0, 1). \]

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If \(R\) has more primes, introduce more height coordinates.
Plücker incidence relations

\[
\min\{t_i(Abc) + t_i(Ad), t_i(Abd) + t_i(Ac), t_i(Acd) + t_i(Ab)\}
\]

is attained twice \hspace{1cm} (D10)

\[
\min\{t_i(Abc) + t_{i+1}(Ad), t_i(Abd) + t_{i+1}(Ac), t_i(Acd) + t_{i+1}(Ab)\}
\]

is attained twice \hspace{1cm} (D01)

are the tropicalizations of incidence relations between \(Gr(|A| + 1, n)\) and \(Gr(|A| + 2, n)\).

If \(L_i(k)\) is the tropical linear space with Plücker coordinates \(t_i(M(A))\) for \(A \in \binom{E}{k}\), then

\[
\begin{align*}
&\text{L}_0(0) \rightarrow \cdots \rightarrow \text{L}_0(k) \rightarrow \text{L}_0(k + 1) \rightarrow \cdots \rightarrow \text{L}_0(n) \\
&\text{L}_1(0) \rightarrow \cdots \rightarrow \text{L}_1(k) \rightarrow \text{L}_1(k + 1) \rightarrow \cdots \rightarrow \text{L}_1(n)
\end{align*}
\]
The parameter space

The **standard flag** of tropical linear spaces has all Plücker coordinates zero.

**Theorem**

*Any diagram* \((L)\) *of tropical linear spaces, in which all Plücker coordinates lie in* \(\text{val}(R)\) *and* \(L_i(\cdot)\) *is the standard flag for* \(i \gg 0\), *corresponds to a matroid over* \(R\).*

Not uniquely, since Plücker coordinates of \(L_i(k)\) are nonunique.
Fix a complete flag $\mathcal{F}$ in $\mathbb{C}^n$ and let $w = s_{i_1} \ldots s_{i_\ell}$ be a word in $A_{n-1}$.

The Bott-Samelson variety of $w$ is

$$Z_w = \{(\mathcal{F}_0, \ldots, \mathcal{F}_s) \in \mathcal{F}\ell_n^{s+1} : \mathcal{F}_0 = \mathcal{F}, \mathcal{F}_k \text{ and } \mathcal{F}_{k+1} \text{ agree except in the } i_k\text{-dimensional space}\}.$$ 

Let $Z_w^{\text{trop}}$ be its naive tropical analogue using Dressians.

**Theorem**

The parameter space of $n$-element matroids over $R$ of global rank $r$ is

$$\lim_{k \to 0} O_k$$

where $O_k$ is a $r(n-r) + kn \text{ dim'}l$ orthant bundle over $Z_{w,c}^{\text{trop}}$, with $w$ a longest Grassmannian word and $c$ a certain Coxeter element.

**Tropical Schubert calculus?**
Fix a complete flag $\mathcal{F}$ in $\mathbb{C}^n$ and let $w = s_{i_1} \ldots s_{i_\ell}$ be a word in $A_{n-1}$. The **Bott-Samelson variety** of $w$ is

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Let $Z_{trop}^w$ be its naive tropical analogue using Dressians.

**Theorem**

The parameter space of $n$-element matroids over $\mathbb{R}$ of global rank $r$ is

$$\lim_{k \to 0} O_k$$

where $O_k$ is a $r(n-r) + kn$ dim’l orthant bundle over $Z_{trop}^{wc_k}$, with $w$ a longest Grassmannian word and $c$ a certain Coxeter element.
Bott-Samelson varieties

Fix a complete flag $\mathcal{F}$ in $\mathbb{C}^n$ and let $w = s_{i_1} \ldots s_{i_\ell}$ be a word in $A_{n-1}$.

The Bott-Samelson variety of $w$ is

$$Z_w = \{(\mathcal{F}_0, \ldots, \mathcal{F}_s) \in \mathcal{F}\ell_n^{s+1} : \mathcal{F}_0 = \mathcal{F},$$

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Tropical Schubert calculus?
Coxeter generalizations?

Bott-Samelson varieties exist in any Weyl type:

$$Z_w = \{(x, e_{i_1}(t_1)x, \ldots, e_{i_\ell}(t_\ell) \cdots e_{i_1}(t_1)x) \}\subseteq (G/B)^{\ell+1}$$

where the $e_i$ are Chevalley generators.

Tropical Dressians should exist too.

Question Does this have anything to do with a valuation ring anymore?

As for $P(M)$, it has edges in directions of the $A_{n+1}$ roots
(in which an $A_1$ orthogonal to the $A_{n-1}$ has a special role.)

Question $A_{n-1} : A_{n+1} :: W : $ what?

Question Are these two connected?

Thank you!
Coxeter generalizations?

Bott-Samelson varieties exist in any Weyl type:

\[ Z_w = \{(x, e_{i_1}(t_1)x, \ldots, e_{i_\ell}(t_\ell) \cdots e_{i_1}(t_1)x)\} \subseteq (G/B)^{\ell+1} \]

where the \( e_i \) are Chevalley generators.

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**Thank you!**
A bit more on DVRs

There is a bijection between finitely generated modules over a DVR & partitions allowing infinite parts.

Example

$N_\lambda = R \oplus R/m^3 \oplus R/m$

$\lambda = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array} \cdots$

Theorem (Hall, …)

The number of exact sequences

$$0 \to N_\lambda \to N_\nu \to N_\mu \to 0$$

up to $\cong$ of sequences is the LR coeff $c_{\lambda\mu}^\nu$ (or its infinite-rows analog).

So, quotients by one element give the Pieri rule.

Lemma, en route to Theorem 3

$M$ is a 1-element matroid over $R \iff M(\emptyset)$ has at most one box more in each column than $M(1)$. 