Valuative invariants for polymatroids

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Outline

- Matroids and polymatroids
- The Tutte polynomial: a motivating example
- Valuations
- Canonical bases for (poly)matroids and valuations
- (Hopf) algebras of valuations
Definition (Edmonds; Gelfand-Goresky-MacPherson-Serganova)

A matroid $M$ (on the ground set $[n]$) is a polytope such that

- every vertex (basis) of $M$ lies in $\{0, 1\}^n$;
- every edge of $M$ is parallel to $e_i - e_j$ for some $i, j \in [n]$. 

\[ \begin{array}{ccc}
100 & 010 & 110 \\
001 & 101 & 011 \\
\end{array} \]
Polymatroids

Definition (Edmonds)

A **polymatroid** $M$ (on $[n]$) is a polytope such that

- every vertex of $M$ lies in $\mathbb{Z}_{\geq 0}^n$;
- every edge of $M$ is parallel to $e_i - e_j$ for some $i, j \in [n]$.

Polymatroids are Postnikov’s (lattice) **generalised permutahedra**.
Let $e_X = \sum_{i \in X} e_i$.
The rank function of $M$ is its support function on 0-1 vectors:

$$\text{rk}_M(X) = \max_{y \in M} \langle y, e_X \rangle.$$ 

**Fact**

0-1 vectors are the only facet normals of (poly)matroids.

$$M = \{ y \in \mathbb{R}^n : \langle y, e_X \rangle \leq \text{rk}_M(X) \ \forall X \subseteq [n], \langle y, e_{[n]} \rangle = \text{rk}_M([n]) \}.$$ 

$r := \text{rk}_M([n])$ is called the rank of $M$. 
A motivating example: the Tutte polynomial

Matroids have two operations yielding *minors*:

- deletion, \( M \setminus i = \{ M \cap x_i = 0 \} \)
- contraction, \( M/i = \{ M \cap x_i = 1 \} \)

Many invariants (e.g. \# bases, independent sets, spanning sets; chromatic and flow polys of graphs; many hyperplane arr. properties; ...) can be evaluated by a *deletion-contraction recurrence*,

\[
f(M) = f(M \setminus i) + f(M/i). \tag{1}
\]

**Theorem (Tutte ’54, Crapo ’69)**

The *Tutte polynomial*

\[
T(M; x, y) = \sum_{X \subseteq [n]} (x - 1)^{r - \text{rk}(X)}(y - 1)^{|X| - \text{rk}(X)}
\]

is universal for (1).

\[\mathbb{Z}\{\text{matroids}\} / (M = M \setminus i + M/i) = \mathbb{Z}[x, y].\]
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A decomposition $\Pi = (P; P_1, \ldots, P_k)$ is a polyhedral complex. We write $P_I = \bigcap_{i \in I} P_i$.

**Example**

$$P_{\{1,2\}} = P_{\{1\}} + P_{\{2\}} - P_\emptyset = P$$

A valuation on a set $\mathcal{M}$ of polyhedra is an $f : \mathcal{M} \rightarrow G$ such that any decomposition $\Pi$ with all $P_I \in \mathcal{M}$ satisfies

$$\sum_{I \subseteq [k]} (-1)^{|I|} f(P_I) = 0.$$
Examples of valuations

\[ \sum_{I \subseteq [k]} (-1)^{|I|} f(P_I) = 0 \]

General examples

- The map \([\cdot]\) sending \(P\) to its indicator function \([P] : \mathbb{R}^n \to \mathbb{Z}\).
  Many interesting evaluations, and sums and integrals of these: volume, Ehrhart polynomial, . . .
- **Euler characteristic** \(\chi\), \(\chi(P) = 1\) for \(P \neq \emptyset\) if \(P\) compact.

From now on \(\mathcal{M} = \{\text{matroids}\}\) or \(\{\text{polymatroids}\}\).

Matroidal examples

- the Tutte polynomial \(T\)
- Speyer’s invariant \(h\), arising from \(K\)-theory of Grassmannians
- Billera-Jia-Reiner’s \(G\), from combinatorial Hopf land
Matroid polytope decompositions come up in

- labelling fine Schubert cells in the Grassmannian (Lafforgue); connections to realisability.
- describing linear spaces via tropical geometry (Speyer, Ardila-Klivans).
- compactifying moduli of hyperplane arrangements (Hacking-Keel-Tevelev).

**Problem**

Describe all (poly)matroid valuations. Find a universal one. Prove [Derksen ’08]’s conjectured universal invariant $G$.

**Notation**

Let $\mathcal{P}_\mathcal{M}$ be the $\mathbb{Z}$-module generated by indicators $[M]$ for $M \in \mathcal{M}$. Grading: $\mathcal{P}_\mathcal{M}(r, n)$ is gen. by rank $r$ matroids on $[n]$.

Prop’n: $\mathcal{P}_\mathcal{M}^\vee := \bigoplus \Hom(\mathcal{P}_\mathcal{M}(r, n), G)$ is the group of valuations.
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Bases

Define the polyhedra (full-dimensional cones)

\[ R(X, r) = \{ y \in \mathbb{R}^n : \langle y, e_{X_i} \rangle \leq r_i \quad \text{for } i = 1, \ldots, \ell - 1, \quad \langle y, e_{[n]} \rangle = r \} \]

and the (almost dual) valuations

\[ s_{X, r}(M) = \begin{cases} 
1 & \text{if } \text{rk}_M(X_i) = r_i \text{ for } i = 1, \ldots, \ell, \\
0 & \text{otherwise}
\end{cases} \]

for \( \emptyset \subsetneq X_1 \subsetneq \cdots \subsetneq X_{\ell-1} \subsetneq X_\ell = [n] \) and \( r = (r_1, \ldots, r_\ell = r) \in \mathbb{Z}^\ell \).

Let \( \Delta_M(r, n) \) be the largest polyhedron in \( M(r, n) \).

Theorem (Derksen-F)

- The distinct nonzero \([R(X, r) \cap \Delta_M(r, n)]\) form a basis for (poly)matroids mod subdivisions \( \mathcal{P}_M(r, n) \).
- The distinct nonzero \( s_{X, r}|_M \) form a basis for valuations \( \mathcal{P}_M^\vee(r, n) \).
Theorem (Brianchon, Gram)

If the polyhedron $P$ does not contain a line, then

$$[P] = \sum_{F} (-1)^{\dim F} [\text{cone}_F(P)]$$

where $F$ runs over all the bounded faces of $P$.

Proposition (Derksen-F)

$$[M] = \sum_{X} (-1)^{n-\ell(X)} [R(X, \text{rk}_M(X))]$$

where $X$ ranges over all chains.
Example of the Brianchon-Gram Theorem

Example

This polytope has the combinatorial type of the permutahedron.

\[
\begin{align*}
    &= + + + \\
    + &+ + + \\
    + &+ - - \\
    - &- - - \\
    + &+ + + \\
\end{align*}
\]
**Theorem (Brianchon, Gram)**

If the polyhedron $P$ does not contain a line, then

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where $F$ runs over all the bounded faces of $P$.

**Proposition (Derksen-F)**

$$[M] = \sum_X (-1)^{n-\ell(X)} [R(X, \text{rk}_M(X))],$$

where $X$ ranges over all chains.
Example

We decompose this polymatroid polytope in $R^s$ by inflating it to the previous one:

\[
\begin{align*}
\triangle & = + \quad \triangle + \quad \triangle + \quad \triangle + \\
\quad + & + \quad \triangle - \quad \triangle - \\
\triangle - & - \quad \triangle - \quad \triangle - \\
\quad + & + \quad \triangle + \quad \triangle + \\
\end{align*}
\]
Define the polyhedra (full-dimensional cones)

\[ R(X, r) = \{ y \in \mathbb{R}^n : \langle y, e_X \rangle \leq r_i \text{ for } i = 1, \ldots, \ell - 1, \langle y, e_n \rangle = r \} \]

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Our bases are unions of $\mathfrak{S}_n$-orbits.

So *unlabelled* (poly)matroids and valuative *invariants* are easy:

**Theorem (Derksen-F)**

- The distinct nonzero $[R(X, r) \cap \Delta_M(r, n)]$ for a fixed maximal chain $X$ form a basis for *unlabelled* (poly)matroids mod subdivs $\mathcal{P}_M(r, n)/\mathfrak{S}_n$.

- The distinct nonzero $\sum_X a$ maximal chain $s_{X, r} | M$ form a basis for *valuative invariants* $\mathcal{P}_M^\vee(r, n)/\mathfrak{S}_n$.

The $R(X, r) \cap \Delta_{Mat}$ are exactly the polytopes of *Schubert matroids*. 
Our bases are unions of $\mathfrak{S}_n$-orbits. So *unlabelled* (poly)matroids and valuative *invariants* are easy:

**Theorem (Derksen-F)**

- The distinct nonzero $[R(X, r) \cap \Delta_M(r, n)]$ for a fixed maximal chain $X$ form a basis for *unlabelled* (poly)mats mod subdivs $P_M(r, n)/\mathfrak{S}_n$.
- The distinct nonzero $\sum_X a$ maximal chain $s_X, r\rvert_M$ form a basis for *valuative invariants* $P_M(r, n)\mathfrak{S}_n$.

The $R(X, r) \cap \Delta_{\text{Mat}}$ are exactly the polytopes of Schubert matroids.
A matroid example

Example

At left: one element $R(X, r) \cap \Delta_{\text{Mat}}$ of the basis of $\mathcal{P}_{\text{Mat}}$ from each $S_4$-orbit, for $(n, r) = (4, 2)$.

$X = \emptyset, 1, 12, 123, 1234.$

\[
\begin{array}{ccc}
0 & 122 & 112 \\
1 & 122 & 112 \\
2 & 012 & 002 \\
3 & 012 & 0112 \\
4 & 012 & 0112 \\
\end{array}
\]

Only one $S_4$-orbit of matroid polytopes isn’t $\Delta_{\text{Mat}} \cap$ a full-dimensional cone:
Hopf algebras of (poly)matroids

$\mathbb{Z}M$, $\mathbb{Z}M/\mathcal{S}_\infty$, $\mathcal{P}_M$, and $\mathcal{P}_M/\mathcal{S}_\infty$, and their duals, are Hopf algebras bigraded by $(n, r)$. The morphisms between them are Hopf too.

- matroids: [(Crapo-)Schmitt]
- polymatroids: [Ardila-Aguiar]

- Product is direct sum of (poly)matroids,
  \[ M_1 \cdot M_2 = M_1 \times M_2 = \{(m_1, m_2) : m_i \in M_i\} \]
- Coproduct is a sum over restrictions and contractions:
  \[ \Delta M = \sum_{X \subseteq [n]} M \setminus ([n] \setminus X) \otimes M/X \]

```
10
\cdot 01
01
\Rightarrow
10 =
0101 0110
```

```
\Delta = \otimes + \otimes + \otimes + \otimes
```
Hopf algebra structure of invariants

Theorem (Derksen-F)

The $\mathbb{Q}$-valued (graded) valuative invariants $(\mathcal{P}_\mathcal{M})^{S_\infty}$ form a free associative algebra:

- $\mathbb{Q}\langle u_0, u_1 \rangle$ for $\mathcal{M} = \{\text{matroids}\}$
- $\mathbb{Q}\langle u_0, u_1, \ldots \rangle$ for $\mathcal{M} = \{\text{polymatroids}\}$.

We’ve reindexed: $u_r = s([1], \ldots, [k]), (r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_k)$.

Then $u_r u_s = u_{rs}$, and each $u_{ri}$ is primitive, $\Delta u_i = u_i \otimes 1 + 1 \otimes u_i$.

As a Hopf alg $\mathbb{Q}\langle u_0, u_1, \ldots \rangle \cong \text{NSym}$ is graded dual to $\text{QSym}$, the Hopf alg of quasisymmetric functions.

We get a double dual map $\mathcal{P}_{\mathcal{M}}/S_n \cong \text{QSym}$:

$$\mathcal{G}(M) = \sum_r u_r(M) u_r^*.$$

Corollary (Derksen’s conjecture)

$\mathcal{G}$ is a universal valuative invariant of (poly)matroids.
**Theorem (Derksen-F)**

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$$G(M) = \sum_R u_{\mathcal{L}}(M) u_{\mathcal{L}}^*.$$  

**Corollary (Derksen’s conjecture)**

$G$ is a universal valuative invariant of (poly)matroids.
Additive invariants

Definition

A valuation $f$ is additive if $f(M) = 0$ whenever $\dim M < n - 1$.

So $f$ adds on top-dimensional pieces in subdivisions.

Theorem (Derksen-F)

The additive valuative invariants form the free Lie alg $\mathbb{Q}\{u_0, u_1, \ldots\}$ whose universal enveloping alg is $(P^\vee_M)^{\mathfrak{S}_n}$.

Some ingredients:

Dimension gives filtrations on our Hopf algebras.

(Poly)matroids are uniquely direct sums of connected (poly)matroids, $M$ on $[n]$ with $\dim M = n - 1$.

$$\text{gr}(P_M/\mathfrak{S}_\infty) = \text{Sym}((P_M/\mathfrak{S}_\infty)_1)$$

Check one containment + enumeration.
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Check one containment + enumeration.
One intriguing future direction:
Knot diagrams can be dualised to yield graphs (i.e. matroids) with their edges (i.e. elements) two-coloured to retain crossing information.
In this setting, some known knot invariants, including the Jones polynomial, appear to become coloured matroid valuations!
Can we get new knot invariants?

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Thanks for your attention!
We get generating functions:

<table>
<thead>
<tr>
<th></th>
<th>(\sum \frac{\dim P(r, n)}{n!} x^n y^r)</th>
<th>(\sum \dim P(r, n)/\mathfrak{S}_n x^n y^r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>matroids</td>
<td>(\frac{xy - y}{xye^{-xy} - ye^{-y}})</td>
<td>(\frac{1}{1 - xy - y})</td>
</tr>
<tr>
<td>polymatroids</td>
<td>(\frac{e^x(1 - y)}{1 - ye^x})</td>
<td>(\frac{1 - x}{1 - x - y})</td>
</tr>
</tbody>
</table>

In fact \(\dim P_{PMat}(n, d)/\mathfrak{S}_n = \binom{n + d - 1}{d}\) and \(\dim P_{Mat}(n, d)/\mathfrak{S}_n = \binom{n}{d}\).
Multiplicative invariants

**Definition**
A function $f : \mathcal{M} \to \mathbb{R}$ is **multiplicative** if $f(M_1)f(M_2) = f(M_1 \oplus M_2)$ for any (poly)matroids $M_1, M_2$.

Thus, $f$ is multiplicative $\iff$ it is a group-like element of $(\mathcal{P}_\mathcal{M})^\mathcal{S}_\infty$.

**Example**
The Tutte polynomial $T(x, y)$ is multiplicative, and

$$T = e^{(y-1)u_0 + u_1} e^{u_0 + (x-1)u_1}.$$