Research statement Alex Fink

My research lies at the intersection of combinatorics, commutative algebra, and algebraic geometry. The main focus of my research is to develop algebraic and algebro-geometric theory to provide unifying characterizations of combinatorial results. In this statement I will discuss a cluster of projects in the following directions:

(1) Matroid invariants as polytope valuations, and from a Hopf-theoretic perspective.
(2) Matroid invariants in K-theory and in representation theory of tensors.
(3) Unifying variants of matroids through commutative algebra.
(4) Moduli spaces in tropical geometry: Grassmannians, Chow varieties, etc.

I will close with short discussions of some research interests that lie outside this cluster.

1. Matroids and valuative invariants

Matroids are structures on finite sets which capture the combinatorics of notions of *dependence*. For example, an application of matroids that was prominent in the first decades after their introduction (independently by Whitney and Maclane) was to graphs, where a set of edges in a graph is dependent if it contains a cycle.

But the paramount example of dependence is linear dependence. Indeed, matroid theory subsumes essentially all the combinatorics of linear algebra, and thus arises naturally when combinatorializing a situation in which linear equations appear. To give a few equivalent views of the basic construction, a matroid can be constructed from a list of vectors, or its dual hyperplane arrangement; or a linear subspace of a vector space with distinguished basis, in which case we use the hyperplane arrangement induced by intersection with the coordinate subspaces.

One definition of matroids which shows off a less linear-algebraic side of the theory is as a class of lattice polytopes. A *matroid (polytope)* on a finite ground set E is a polytope in $\mathbb{R}^E$ whose vertices lie in $\{0,1\}^E$ and whose edges are parallel translates of $e_i - e_j$ for some $i, j \in E$. For instance, given a vector configuration $\mathbf{v}$, the vertices of its matroid polytope are the $\{0,1\}$ indicator vectors of subsets of $\mathbf{v}$ which are bases for the ambient space.

Matroid polytopes have shown up in several algebro-geometric contexts: as one famous example, they appear in Lafforgue's work on the Langlands program [28]. Two fundamental facts underlie these applications. The Grassmannian $G_r(\mathbb{K}^n)$ of r-planes in n-space, for $0 \le r \le n$, carries an action of the torus $(\mathbb{K}^*)^n$, and orbit closures X under this action are toric varieties whose moment polytopes are matroid polytopes. If X degenerates torically to a reducible scheme Y, then the components of Y are also torus orbit closures, and their moment polytopes form a cell complex whose total space is the moment polytope of X. The intersections of the components of Y have moment polytopes which are the intersections of the corresponding faces in this cell complex, This motivates understanding *subdivisions* of matroid polytopes, which are polyhedral complexes every face of which is a matroid.

A number of matroid functions of interest, including the much-loved Tutte polynomial [7, 8] and the more recent examples introduced in [4, 12, 34], are *valuations* on matroid polytopes. A valuation f is a function on subsets of a space that factors as $f = f' \circ \chi$, where χ sends a set to its indicator function and f' is linear. Thus, if f is valuative, given any polytopal subdivision, there is an inclusion-exclusion additive relation among the values of f on the faces of the subdivision.

In work with Harm Derksen [13], we provide a combinatorial basis for the valuative matroid functions, (and dually, the space spanned by indicator functions of matroid polytopes.) Here is the snappiest variant of this result:

Theorem 1.1 ([13], Thm 6.3)**.** *The vector space spanned by indicator functions of all matroids, modulo the $\mathfrak{S}_n$-action, has a basis consisting of the* Schubert matroids*.*

A Schubert matroid is the matroid of a generic point in a Schubert cell within the Grassmannian (that is, of a linear subspace with pivots of its row echelon form in specified columns). The proof uses the Brianchon-Gram theorem, which writes the indicator function of a polytope as the sum of indicator functions of its tangent cones. As a corollary, we prove a conjecture of Derksen in [12] that an invariant $\mathcal{G}$ there defined specializes to every valuative invariant. This $\mathcal{G}$ was introduced as a generalization of an invariant of Billera, Jia, and Reiner [4] where also the connection between Hopf algebras and valuativity was first noted.

Matroid valuations also form a Hopf algebra structure, dual to the Hopf algebra of the indicator functions introduced by Crapo and Schmitt (e.g. [9]). Derksen and I show that passing to *polymatroids* yields familiar Hopf algebras here. A polymatroid is to a subspace arrangement what a matroid is to a vector arrangement; a polymatroid polytope is a matroid polytope without the $\{0, 1\}$ restriction on the vertices.

Theorem 1.2 ([13], Thm 1.7)**.** *As a combinatorial Hopf algebra, the $\mathfrak{S}_n$-invariant polymatroid valuations are isomorphic to* NSym*, the Hopf algebra of noncommuting symmetric functions.*

The Tutte polynomial of a matroid has a simple expression in the Hopf algebra, which allows a generalization to polymatroids, although the result is a rational function. Independently, certain evaluations of the Tutte polynomial have been generalized to polymatroids by Kálmán [26], following the classical formula in terms of activity. These do not agree where they are both defined.

Question 1.1. Can the above two definitions of the Tutte polynomial for polymatroids be reconciled?

It is also interesting that noncommuting symmetric functions are not manifest in the description of Theorem 1.2. I intend to consider the following question of Pasha Pylyasvkyy:

Question 1.2. How are the more familiar bases for the Hopf algebras of polymatroid valuations to be interpreted in terms of matroid combinatorics?

For example, the multiplication rule in NSym of Section 8 of [29] has a K-theoretic flavor; it may be interpretable in terms of the K-theoretic invariant discussed below.

2. Matroids and K-theory

The algebraic geometry of vector arrangements and subspaces straightaway gives rise to matroids. As above, given a torus orbit closure X on a Grassmannian $G_r(\mathbb{C}^n)$ and a torus-equivariant flat degeneration Y of X, the way components of Y intersect is encoded by a matroid polytope subdivision. The equivariant K-class of Y may be computed as an alternating sum over the classes of these intersections of components. Also, equivariant K-class is preserved by flat degenerations.

This is where my work with David Speyer (U. Michigan) in [21] begins. From the above facts we are close to being able to conclude the following:

Theorem 2.1 ([21], §3)**.** *There is a matroid valuation y, valued in the T-equivariant K-theory ring $K_T^0(G_r(\mathbb{C}^n))$, such that if X is an orbit closure with matroid polytope M, then $y(M)$ is the K-class of X.*

The remaining ingredients are, first, that two orbits with the same matroid have the same K-class; second, that it is possible to assign K-classes to matroids that correspond to no orbit. The construction proving these uses equivariant localization. A class in $K_T^0(G_r(\mathbb{C}^n))$ is determined by its restriction to each torus fixed point x, which is the multigraded Hilbert functions of the class restricted to the affine cell near x. By toric geometry, these Hilbert functions are lattice point enumerators of tangent cones, so they depend only on, and can be defined from, the polyhedron. The computation again involves, more or less, the Brianchon-Gram theorem.

My main theorems with Speyer relate to evaluations of the invariant y of Theorem 2.1.

Theorem 2.2 ([21], Thms 5.1, 6.1, 6.5)**.** *Both the Tutte polynomial and Speyer's tropical face enumerator invariant h (introduced in* [34]*) are evaluations of y.*

In fact, these two invariants arise in strikingly similar fashion. One can pull and push K-classes from the Grassmannian to the partial flag variety $\mathcal{F}$ of 1- and $(n-1)$-dimensional subspaces of $\mathbb{C}^n$, via a larger partial flag variety mapping to both. The K-theory of $\mathcal{F}$ is generated as a ring by the classes α and β of its two Schubert divisors. Unadorned, this push-pull operation yields an evaluation of Speyer's h (a univariate polynomial), whereas first twisting by $\mathcal{O}(1)$ on the Grassmannian produces the Tutte polynomial in α and β.

Speyer and I intend to continue this project. For one, more familiar matroid theory may be recoverable: e.g. can one recover from y the activity formula for Tutte using discrete Morse theory? We will also apply geometric positivity results in this setting:

Question 2.1. What do geometric results about positivity in K-theory tell us about matroids? Can we prove, e.g., the conjecture on the convexity of evaluations of the Tutte polynomial at $x + y = p$ (see [30]), or Speyer's conjectured bounds on h?

A step in a similar direction is my result with Andrew Berget giving a positive expression for the *ordinary cohomology* class (which is coarser than the equivariant K-class) of the orbit for a uniform matroid. Klyachko [27] has given a positive formula for this class as well, of quite different appearance.

Further afield, an exciting recent development, without direct relation to [21] but of a similar flavor, is Huh's proof [25] of the long open conjecture that the chromatic polynomial of a graph is log-concave. Huh's proof is via complex singularity theory. Eric Katz has reinterpreted the proof using tropical geometry on the graph of the standard Cremona transform. The same variety arises in [32], where its K-theory is connected to broken circuits in matroids.

Question 2.2. Can chromatic polynomials of matroids be proven log-concave by integrating techniques of [25, 32]?

In ongoing joint work with Andrew Berget, I am investigating the similar situation where instead of considering orbits of the torus $(\mathbb{C}^*)^n$ in $G_r(\mathbb{C}^n)$, one "un-quotients" by GL_r and takes a $\mathrm{GL}_r \times (\mathbb{C}^*)^n$ orbit within the affine space of $r \times n$ matrices. The motivation is to understand the representation theory of the *tensor module*: this is the representation spanned by a GL_r orbit within $(\mathbb{C}^r)^{\otimes n}$. Indeed, the tensor module is a sum of some graded components of the multigraded Hilbert function of the $\mathrm{GL}_r \times (\mathbb{C}^*)^n$ orbit.

Decomposing the tensor module is related to the problem of when symmetrizations of decomposable tensors are zero, a problem studied by Dias da Silva's school. This program might yield insight into some famous questions on vector configurations such as Rota's basis conjecture [24], in which there has been a recent resurgence of interest on account of geometric complexity theory.

The nearest big question in this line of inquiry remains open:

Question 2.3. Does the equivariant K-class of a $\mathrm{GL}_r \times (\mathbb{C}^*)^n$ orbit of $r \times n$ matrices depend only on the matroid of a generic point?

We have partial results, for the extremal nontrivial cases of the problem (recall that $0 \leq r \leq n$), as well as for certain coefficients in classes of arbitrary orbits.

Theorem 2.3. *The answer to Question 2.3 is yes when $r = 2$ or $r = n - 2$.*

Theorem 2.4. *The coefficients of the equivariant K-class in Question 2.3 corresponding to hooks in the GL_r variables and squarefree monomials in the torus characters (essentially) enumerate dependent sets of the matroid.*

The obstacle to lifting the result sought in Question 2.3 directly from the Grassmannian is the existence of matrices of smaller rank than r, which do not map to points of the Grassmannian. Berget and I can characterize the orbits appearing in this lower-rank locus. One might hope to understand this locus with inductive techniques, i.e. decomposing it by intersecting with divisors. Naïve attempts at this reach the obstacle that components of the intersection may be larger than single orbits, but more careful choices, or broadening the class of varieties handled, may be possible. Our $\mathrm{GL}_r \times (\mathbb{C}^*)^n$ orbits are also varieties of representations for a directed $K_{n,1}$ quiver, and the theory of these may provide insight.

One alternate approach to Question 2.3 which seems promising invokes a partial desingularization π of an $\mathrm{GL}_r \times (\mathbb{C}^*)^n$ orbit X of matrices, which is simultaneously a vector bundle over a torus orbit on the Grassmannian. If it could be shown that the higher derived pushforwards of the structure sheaf along π vanished, we could relate K-classes on X to classes on the desingularization and therefore to classes on the Grassmannian; as a bonus, we would show X to have rational singularities. This thus reduces to showing cohomological vanishing for a collection of toric vector bundles.

Lastly, given my interest in toric varieties associated to matroids, it is natural to bear the following conjecture of White in mind. It relates to the toric ideal of our torus orbits, that is, the kernel of the monomial map whose monomials are the bases of the matroid.

Question 2.4. Is the toric ideal of a matroid generated by quadrics?

3. Unifying variants of matroids

A number of enriched variants of matroids have been defined, retaining information about a vector configuration (or equivalent object) richer than the purely linear-algebraic information that matroids retain. It is of interest to generalize matroid combinatorics to these settings, some of which are oriented matroids [5], valuated matroids [14], complex matroids [2], and arithmetic matroids [10]. For example, an arithmetic matroid endows a matroid with the extra data of the cardinalities of certain finite abelian groups derived from an integer vector configuration. Remembering this arithmetic data is useful to produce objects like a toric arrangement, a partition function, and a zonotope (see [11]).

In order to achieve this, it is natural to attempt to unify these generalizations of matroids under a common framework. For instance, one such unification was sought by Dress in his program of matroids with coefficients (represented in [14]). In work with Luca Moci [20], we suggest a different unification by defining the notion of a *matroid M over a commutative ring R*. Such an M assigns, to every subset A of the ground set, a finitely generated R-module $M(A)$. The axiom we impose is that every two-element minor of M must arise from a two-element vector configuration in an R-module.

Theorem 3.1 ([20], Prop 2.9, Cors 5.7, 6.3)**.**

(1) *If R is a field, matroids over R are equivalent to matroids.*
(2) *If R is a DVR, a matroid over R contains all the data of a valuated matroid.*
(3) *If $R = \mathbb{Z}$, a matroid over R contains all the data of an arithmetic matroid.*

Moci and I also generalize the Tutte polynomial, viewing it, as [7] does, as the map to the so-called *Tutte-Grothendieck* ring, a ring generated by classes of matroids with some relations imposed on sums and products.

As the definition of matroids over a ring is new, there are multiple expansions to be made; fortunately, the definitions use sufficiently standard objects that there likely remains some low-hanging fruit to be obtained by standard commutative-algebraic or categorical techniques. First, even the simplest operations characteristic of matroids are so far defined only for regular rings of dimension at most one.

Question 3.1. Extend duality of matroids over rings, the Tutte-Grothendieck computation, etc., to non-regular rings and rings of dimension ≥ 2.

A seemingly key property of one-dimensional rings which fails in higher dimension is, loosely, that if one knows the R-modules M and $M/\langle x\rangle$ up to isomorphism, then one also knows $x \in M$ up to isomorphism. Thus, matroids over higher-dimensional rings may benefit from being decorated with extra data which will recover the analogue of this property.

Some types of decorated matroids are not captured by our construction:

Question 3.2. Can oriented and complex matroids be understood as generalizations of matroids over rings?

A category more general than modules over a ring may be required. Alternatively, it may be necessary to generalize the category-theoretic underpinnings. For instance, two-element matroids over rings are certain commutative squares; oriented matroids could be expected to be better behaved in a context where squares were expected to anticommute.

A further project in this direction is to generalize the algebraic geometry related to polytopes and their subdivisions, described above, to these classes of matroids. A priori, base-changing the ground field of a variety to another ring is something to which algebraic geometry is eminently suited. However, this change of base does not change the equivariant K-theory in the same fashion as our Tutte-Grothendieck invariant changes, so modifications will be needed to recover this. Hinting at what may be involved, there is the fact that the polytope subdivisions that determine tropical objects (see below) introduce one more dimension, corresponding to valuations of field elements.

As one more promising direction, a better structural understanding of matroids over DVRs may provide the needed impetus to give a combinatorial classification of point configurations according to their disposition within the Bruhat-Tits building. Sturmfels suggests that this project may allow a better understanding of tropical convex hulls, and thereby

much of tropical linear algebra. Quantities like valuations of minors of a matrix would arise in this classification, and matroids over a DVR record these.

4. Tropical geometry

Tropical geometry has blossomed in the decade and a half since its introduction. At its core it is a combinatorialization of ("classical") algebraic geometry, associating to each algebraic variety a combinatorial "shadow" in the form of a weighted polyhedral complex. Tropical geometry has been dramatically effective for, e.g., enumerative problems, such as finding explicit recursions for a class of Gromov-Witten invariants in the plane [6]. Parallelling the historical development of algebraic geometry, the situation in which tropical geometry is now best understood is for varieties embedded in an ambient projective space (or, at least, a toric variety), as opposed to as self-contained abstract objects.

Suppose we have a subvariety of $\mathbb{P}^n$ over an algebraically clsoed field bearing a nontrivial valuation. As a set, its *tropicalization* is simply its image under the coordinatewise valuation map. This procedure can be used as well for trivially valued (i.e. non-valued) fields, by first extending to a field with nontrivial valuation; in this case, the tropicalization is a fan over the origin. However, there is an intrinsic, combinatorial, definition of which polyhedral complexes are tropical varieties, and not all of them arise as tropicalizations (though most do if dimension, codimension, and degree are small).

For example, the tropicalization of a linear subspace X in $\mathbb{P}^n$, over a field with trivial valuation, is determined by its matroid — as might be expected, given that the set of allowable coordinatewise valuations of points on X encodes only which sets of coordinates may become zero simultaneously. When the valuation is nontrivial, then the valuations of minors of X give a height function on the vertices of the matroid polytope, which induces a regular polytope subdivision. The tropicalization of X is determined by this subdivision: it is the union of the cones normal to certain faces. (This is the source of Speyer's invariant h in Theorem 2.2: it enumerates the faces that remain, weighted by $t^{\text{dimension}}$.)

The subtlety of tropical geometry begins to come to the fore in moduli problems. To continue with linear spaces, there is a well-behaved tropical Grassmannian [35], which is the tropicalization of the classical Grassmannian in its Plücker embedding, and is a parameter space for linear spaces that are tropicalizations. The parameter space for all tropical linear spaces is a larger object, named the *Dressian*, which is not even a tropical variety itself.

In ongoing work with Felipe Rincón, I am investigating such a subtle difference in the opposite direction. Tropically, the map from a matrix to its vector of maximal minors, which we call the *Stiefel map*, does not surject onto the Grassmannian, whereas the classical analogue does. Using combinatorics of a certain Newton polytope described in [36], we prove a conjecture from that work on the dimensions of its faces, and obtain:

Theorem 4.1. *The image of the tropical Stiefel map is covered by patches, each of which is bijective on an open set with a coordinate subspace of the domain. The faces of each patch are in bijection with certain classes of tropical hyperplane arrangements.*

This work can be expected to add another angle to the rich combinatorics of tropical hyperplane arrangements and oriented matroids [3], and provide a worked concrete case of Mikhalkin's theory of tropical modifications. We also expect to be able to extract topological information about the Stiefel image, which we can do at present in the rank 2 case.

Turning to moduli problems of more general tropical subvarieties, construction of a tropical Hilbert scheme has been attempted [1]. I hold, though, that this is not the most useful

object to construct. Despite their name, tropical varieties are more like algebraic cycles, i.e. classes in Chow (intersection-theoretic) cohomology, than subvarieties. The weights in a tropical variety retain multiplicities of top-dimensional components, but nowhere is data retained corresponding to embedded components. Accordingly, the natural parameter space is the *Chow variety*, parametrizing cycles of given codimension and (multi)degree.

My work so far in this direction has been to establish the combinatorics of these higher-degree cycles. For a degree 1 cycle X (i.e. a linear space), the matroid polytope is the *Chow polytope*, the weight polytope of the point X in the Chow variety. Similarly,

Theorem 4.2 ([17], Thm 5.1)**.** *Every tropical cycle in a toric variety has a Chow polytope subdivision. It may be obtained by a Minkowski sum calculation.*

The existence of Chow polytopes is straightforward for tropicalizations but not obvious a priori for general tropical varieties. Unfortunately, the Chow polytope subdivision does not determine the combinatorics of the tropical variety in general.

An obvious next problem which I will investigate in this direction is the following:

Question 4.1. Construct the tropical Chow variety, as well as the Chow analogue of the Dressian.

Understanding the difference between these two objects has the potential to speak to a great number of tropical *lifting* problems, i.e. problems asking which tropical varieties are tropicalizations. I expect that the failure of the Chow polytope to determine a tropical variety corresponds to the existence of multiple distinct monomials in the Plücker coordinates of the same torus weight within the coordinate ring of the Grassmannian, and that re-embedding along a linear subspace determined by the set of all bracket monomials will yield a tropical Chow variety that does parametrize tropical cycles.

I will close with the mention of a problem of interest to me in tropical geometry which does not represent a prior direction of substantial research.

Question 4.2. Is there a useful approach to tropical enumerative problems via Euler characteristics, or similar valuation-like invariants, of families?

For example, one classical technique for counting curves with one node in a universal family whose other members are smooth is computing the Euler characteristic of the family: since curves with one node and smooth curves have different Euler characteristics, the number of the former can be extracted. Tropically this argument may be emulable directly, calculating the Euler characteristic from combinatorics of the f-vector of the tropical family. Alternatively, volume (or some other valuative function) may work well in place of Euler characteristic. For instance, the method of "floor diagrams" [6] uses points in the tropical plane at great vertical separation. The volumes in the universal family in question should be linear in each of these separations, call them $\mathbf{s}$, so that enumerative questions should reduce to computing a leading coefficient in $\mathbf{s}$ of the volume, which may be more approachable than finding the volume outright.

5. Other research

In this section I will briefly sketch several other projects. They exemplify a couple important trends in modern research beyond those represented above, and it is also my hope that taken together, these (and the work above) will provide me a broad ground for interfacing with undergraduate students.

My work in [18] is on primary decomposition of an ideal arising from algebraic statistics. Algebraic statistics, in the sense of [15], is founded on the observation that many models of interest to statisticians are in fact algebraic varieties familiar in other contexts: for example, the condition that two discrete random variables be independent says that the vector of real numbers that is their joint probability distribution should lie on the Segre embedding of the product of two projective spaces. Thus, if one has a collection of independence-type conditions — in my example they are *conditional* independence conditions — and wishes to understand which other independence conditions they imply, one can answer the question by describing the irreducible components of an intersection of varieties. The main theorem in [18] does this, constructing a set of graphs in bijection with the components. Indeed, this work kicked off a series of papers generalizing to more ideals, e.g. [23]. Still, analogous questions for many statistical varieties remain open.

The paper [19] belongs in the broad program of associating combinatorial phenomena to one family of Weyl or Coxeter groups, typically type A, and generalizing to the other types. In this case we are dealing with the Catalan numbers, famed for the many enumerative problems they answer [33]. Some of the objects enumerated, like nonnesting and noncrossing partitions (two kinds of set partition of $\{1, \ldots, n\}$, each with some forbidden four-element subpartitions) have natural interpretations in terms of the groups A_n, respectively as antichains of roots and as elements less than or equal to a given Coxeter element in the absolute order. One observes that the cardinalities of these two sets remain equal for any Weyl group. My work with Iriarte Giraldo in [19] produces a bijection between these two sets for the classical groups, which preserves the statistic analogous to block sizes of the partitions. Other variations have also been found, such as a bijection that is uniform over the type of Weyl group considered, and one which shows a cyclic sieving phenomenon. But the Catalan combinatorics is wide, and has intriguing connections to areas such as cluster algebras; there is much more to be asked here.

Finally, [16] belongs among algebraic and algorithmic approaches to combinatorial game theory. Combinatorial game theory, whose objects of interest are two-player complete information games without chance, still mostly lies in the fringes outside serious combinatorics, even though it circumscribes powerful theories for several classes of games. One class, the *misère* games (where a player wins who has no moves remaining) remained little understood until recently (e.g. [31]) with the insight to examine quotient monoids of a certain monoid of game positions. In an attempt to rein in these quotient monoids, Guo and Miller [22] introduced *lattice games*, a framework containing many standard games and suitable for the application of techniques of polyhedral combinatorics and generating functions. They conjectured that the generating function of the winning positions of any lattice game was rational. In [16] I showed this false: lattice games are capable of universal computation, which is a very strong obstacle to any structural assertion one might wish to make about them as a class. The natural next problem is thus to construct a subclass of lattice games which still includes the standard games of interest but where stronger structural facts are true (and there are candidates, e.g. the "square-free" lattice games).

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