POLYMATROID SUBDIVISION

ALEX FINK

These notes are a draft exposition, written for a portion of Jack Edmonds' mini-course *Existential Polytime and Polyhedral Combinatorics* at the London Taught Course Center in June 2015. I thank Jack for inviting me to give part of the course and for helping to shape the direction of these notes.

1. (Poly)matroids and their polytopes

A *polymatroid rank function* on a finite ground set E is a nondecreasing function $\mathrm{rk} : 2^E \to \mathbb{R}$ satisfying

(0) $\mathrm{rk}(\emptyset) = 0$;
(2) $\mathrm{rk}(A) + \mathrm{rk}(B) \geq \mathrm{rk}(A \cap B) + \mathrm{rk}(A \cup B)$ for all $A, B \subseteq E$ (the *submodular inequality*).

(Note that everywhere $\mathbb{R}$ appears in these notes, any ordered abelian group would do just as well, though if the group is not divisible, some of the integrality statements become unimpressive and we need to be more careful about defining subdivisions.) If moreover rk takes only integer values and

(1) $\mathrm{rk}(A) \leq |A|$ for all $A \subseteq E$

then rk is a *matroid* rank function.

Define the *independence polytope* of rk to be

$$P_{\mathrm{ind}}(\mathrm{rk}) = \{x \in \mathbb{R}^E : x_i \geq 0 \text{ for all } i \in E, \sum_{i \in A} x_i \leq \mathrm{rk}(A) \text{ for all } A \in 2^E\}$$

and its *base polytope* to be

$$P_{\mathrm{base}}(\mathrm{rk}) = \{x \in \mathbb{R}^E : \sum_{e \in A} x_e \leq \mathrm{rk}(A) \text{ for all } A \in 2^E, \sum_{i \in E} x_i = \mathrm{rk}(E)\}.$$

If rk is the rank function of a matroid M then this defines the matroid independence polytope $P_{\mathrm{ind}}(M)$ and the matroid base polytope $P_{\mathrm{base}}(M)$.

When Edmonds introduced polymatroids, he defined the *polymatroid* itself to be the polytope $P_{\mathrm{ind}}(\mathrm{rk})$. That terminology is fine, but in these notes I am comparing matroids and polymatroids and contrasting base and independence polytopes, and to speak of P_{ind} will make drawing those distinctions clearest.

Remark. The rank function axioms can alternatively be set up "locally" on the Boolean lattice. If $\mathrm{rk} : 2^E \to \mathbb{R}$ is a function with

(0) $\mathrm{rk}(\emptyset) = 0$;
(2) $\mathrm{rk}(S \cup \{a\}) + \mathrm{rk}(S \cup \{b\}) \geq \mathrm{rk}(S) + \mathrm{rk}(S \cup \{a, b\})$ for all $S \subseteq E$ and $a, b \notin S$

then rk is a polymatroid rank function, and if moreover rk is integer-valued and

(1) $\mathrm{rk}(\{a\}) \leq 1$ for every $a \in S$

then rk is a matroid rank function. The converses are clear.

The first thing that makes the base and independence polytopes worthwhile is that their vertices are well-behaved.

Theorem 1.1. *The vertices of* $P_{\text{ind}}(\text{rk})$ *are*

$$\sum_{i=1}^{k}(\text{rk}(\{a_1,\ldots,a_i\}) - \text{rk}(\{a_1,\ldots,a_{i-1}\}))e_{a_i}$$

as $a_1,\ldots,a_k$ *range over all sequences of distinct elements in* E *(not necessarily exhausting* E*; indeed, possibly empty). The vertices of* $P_{\text{base}}(\text{rk})$ *are the vertices among these where* $\{a_1,\ldots,a_k\} = E$.

In particular, any vertex of a matroid base or independence polytope has all coordinates 0 or 1.

Jack has presented a proof of a form of this theorem, so I omit it. It appears for polymatroids as Theorem 22 of [5].

The vertices of the independence polytope or base polytope of a matroid are the independent sets, respectively bases, of the matroid. This follows from the classical argument that the rank function definition of a matroid is equivalent to the independent set or basis definitions. A good reference for matroids sympathetic to our present point of view is Schrijver's volume [15]. I also recommend the standard references [17, 13], though their perspectives are complementary, e.g. in that they are unconcerned with polytopes.

Corollary 1.2. *Every inequality in the definitions of* $P_{\text{ind}}(\text{rk})$ *and* $P_{\text{base}}(\text{rk})$ *is an equality at some point of the polytope.*

Proof. For the inequalities $x_i \geq 0$ of the independence polytope, take the origin. Otherwise the inequality concerns $\sum_{i\in A} x_i$; take any sequence of elements of E of which some prefix is the list of all elements of A and apply Theorem 1.1. □

Both polytopes contain the same information; it is easy to pass back and forth between them.

Proposition 1.3. *For any (poly)matroid* rk, $P_{\text{base}}(\text{rk})$ *is the face of* $P_{\text{ind}}(\text{rk})$ *maximising the sum of the coordinates. Conversely,*

$$P_{\text{ind}}(\text{rk}) = (P_{\text{base}}(\text{rk}) + \operatorname{cone} -e_i : i \in E) \cap \operatorname{cone}\{e_i : i \in E\}.$$

Here $+$ is Minkowski sum, and $\operatorname{cone} S$ is the set of nonnegative linear combinations of vectors in the set S.

Proof. These are easy once we observe that $P_{\text{base}}(\text{rk})$ lies in the nonnegative orthant: the defining inequalities give $x_e \geq \text{rk}(E) - \text{rk}(E \setminus A)$, which is nonnegative. Adding these nonnegativity conditions to the presentation of $P_{\text{base}}(\text{rk})$, the only difference between them is then the extra equality $\sum_{i\in E} x_i = \text{rk}(E)$ in the latter, which implies the first statement of the proposition. For the second direction, the $\supseteq$ containment is clear because decreasing the x_i (while leaving them positive) makes the inequalities of $P_{\text{ind}}(\text{rk})$ further from tight. For the other direction, if x is a point of the independence polytope, then a new polymatroid rank function rk' is defined by $\text{rk}'(A) = \text{rk}(A) \setminus \sum_{i\in A} x_i$. Choose any vertex x' in $P_{\text{ind}}(\text{rk}')$ that maximises the sum of the coordinates, i.e. where this sum attains the value $\mathrm{rk}'(E) = \mathrm{rk}(E) - \sum_{i\in E} x_i$. Then $x + x'$ is contained in $P_{\mathrm{ind}}(\mathrm{rk})$ and the sum of its coordinates is $\mathrm{rk}(E)$, so that in fact it's contained in $P_{\mathrm{base}}(\mathrm{rk})$. □

Example 1.4. A *permutohedron* (or "permutahedron") is the convex hull of all permutations of the coordinates of a single vector. The permutohedron is useful in combinatorics of finite sets and symmetric groups, especially from the reflection group perspective: for instance, unless the vector has equal coordinates, its poset of faces is isomorphic to the poset of ordered set partitions under refinement, and its 1-skeleton is the Cayley graph of the symmetric group generated by adjacent transpositions.

If all of the coordinates of the vector are positive, say $a_1 \geq \cdots \geq a_n \geq 0$, then the rank function $\mathrm{rk}(A) = a_1 + \cdots + a_{|A|}$ defines a polymatroid (check this yourself!) whose base polytope, by Theorem 1.1, is the permutohedron. In fact, the class of polymatroid base polytopes has been studied in the enumerative tradition of combinatorics under the name of *generalised permutohedra*, starting with Postnikov [14]. The statement above about the poset ordered set partitions should be compared to Proposition 3.2, which implies a surjective poset morphism from this poset onto the faces of any polymatroid base polytope; the difference is that it need no longer be an injection.

2. Matroid union

Theorem 2.1 (Matroid Partition Theorem)**.** *For any set M_i of matroids on the same ground set E, and any set $J \in 2^E$, exactly one of the following is true: either J is partitionable into sets J_i which are independent respectively in M_i, or there is a subset H of J such that $|H| > \sum \mathrm{rk}_i(H)$.*

The partitionable sets J are the independent sets of a matroid M, called the union *of the M_i.*

Proof. "Not both" is easy: if both were true, then

$$\sum_i r_i(H) < |H| = \sum_i |J_i \cap H| \leq \sum_i \mathrm{rk}_i(H).$$

To prove "one or the other", we can give an algorithm to construct whichever one exists. Jack has explained this, so I won't repeat it here. □

The *Minkowski sum* of a collection of polyhedra $P_1, \ldots, P_k$ is defined as

$$P_1 + \cdots + P_k = \{x_1 + \cdots + x_k : x_i \in P_i\}.$$

Proposition 2.2. *Let the polyhedron P_i be the solution set of the system of linear inequalities $Ax \leq b_i$, for $i = 1, \ldots, k$, and suppose that each inequality is tight at some point of P_i for each i. Then the Minkowski sum of the P_i is the solution set of $Ax \leq \sum_{i=1}^k b_i$.*

[[proof? Schrijver? also that submodulars are tight]]

This translates readily to the context of polymatroid polytopes, using Corollary 1.2.

Corollary 2.3. *Let* $\mathrm{rk}_1, \ldots, \mathrm{rk}_k$ *be polymatroid rank functions, and* rk *their sum. Then the independence polytope of* rk *is the Minkowski sum of the independence polytopes of the* rk_i*, and the same is true for base polytopes.*

This provides us a useful restatement of the theorem on matroid unions.

Corollary 2.4. *Let* M *be the union of the matroids* M_i*. Then* $P_{\mathrm{ind}}(M)$ *is the intersection of the Minkowski sum of the* $P_{\mathrm{ind}}(M_i)$ *with the unit cube.*

Proof. Let P be the intersection of the Minkowski sum of the $P_{\mathrm{ind}}(M_i)$ with the unit cube. We first show an integer point is contained in $P_{\mathrm{ind}}(M)$ if and only if it is contained in P. For an integer point e_J of P, partition J into sets J_i independent in M_i. Then $e_{J_i} \in P_{\mathrm{ind}}(M_i)$ for all i, so $e_J = \sum_i e_{J_i}$ is in their Minkowski sum, and is manifestly also in the unit cube. Of the integer points not in $P_{\mathrm{ind}}(M)$, those with a negative coordinate or a coordinate exceeding 1 are clearly not in P. Otherwise we are dealing with a point e_J where J is not a partitionable set, and Theorem 2.1 hands us a functional $\sum_{i\in H} x_i$ whose value on the Minkowski sum of the $P_{\mathrm{ind}}(M_i)$ is bounded above by $\mathrm{rk}_1(H) + \cdots + \mathrm{rk}_k(H) < |H|$, which is less than $\sum_{i\in H} e_J = |H|$.

So we are done as long as all vertices of P are integral. For this we state the rank function rk of the matroid union, in terms of the rank functions rk_i of M_i:

$$\mathrm{rk}(J) = \min_{I\subseteq J} \left(|J \setminus I| + \sum_{i=1}^{k} \mathrm{rk}_i(I) \right).$$

Clearly the inequalities defining P, which by Corollary 2.3 are

$$0 \le x_j \le 1 \text{ for all } j; \qquad \sum_{j\in J} x_j \le \sum_{i=1}^{k} \mathrm{rk}_i(J)$$

are among those defining $P_{\mathrm{ind}}(\mathrm{rk})$: the former come from $I = J = \{j\}$ and the latter from the given choice of J with $I = \emptyset$. Conversely, all of these inequalities are implied by those defining P: sum the former inequality for $J \setminus I$ with the latter for I. So $P_{\mathrm{ind}}(\mathrm{rk}) = P$. But rk can be checked to be a submodular function: given $A, B \subseteq E$, let H and I be the argmin in $\mathrm{rk}(A)$ and $\mathrm{rk}(B)$ respectively; then

$$\begin{aligned}
&\mathrm{rk}(A\cap B) + \mathrm{rk}(A\cup B) \\
&\le \left(|(A\setminus H)\cap(B\setminus I)| + \sum_i \mathrm{rk}_i(H\cap I) \right) + \left(|(A\setminus H)\cup(B\setminus I)| + \sum_i \mathrm{rk}_i(H\cup I) \right) \\
&\le \left(|(A\setminus H)| + \sum_i \mathrm{rk}_i(H) \right) + \left(|(B\setminus I)| + \sum_i \mathrm{rk}_i(I) \right) \\
&= \mathrm{rk}(A) + \mathrm{rk}(B).
\end{aligned}$$

So $P_{\mathrm{ind}}(\mathrm{rk})$ has integer vertices by the greedy algorithm total dual integrality argument. □

To get the analogue of Corollary 2.4 for base polytopes we must introduce a further condition, to deal with the fact that the maximum of the functional $\sum_{i=1}^{n} x_i$ may decrease on intersecting with the cube.

Corollary 2.5. *Let $M_1, \dots, M_k$ be matroids on ground set $[n]$, and suppose that the rank of their union M is the sum of the ranks of $M_1, \dots, M_k$: equivalently suppose that there is a family of bases B_i of M_i for $i = 1, \dots, k$ that are pairwise disjoint. Then*

$$P_{\text{base}}(M) = \left(\sum_{i=1}^{k} P_{\text{base}}(M_i) \right) \cap [0,1]^n.$$

3. Faces of matroid polytopes

In this section we discuss the faces of matroid polytopes, with special attention to those of lowest and highest dimensions. But before that it makes sense to discuss dimensions of (poly)matroid polytopes themselves.

For independence polytopes the answer is simple. If rk is a polymatroid on a ground set E, the dimension of $P_{\text{ind}}(\text{rk})$ is $|E|$ minus the number of elements $i \in E$ with $\text{rk}(i) = 0$. If $\text{rk}(i) = 0$ then $P_{\text{ind}}(\text{rk})$ is contained in the hyperplane $x_i = 0$, and all these hyperplanes are transverse, so the dimension is at most $|E|$ minus the number of these; on the other hand it contains a simplex $\text{conv}(\{0\} \cup \{\epsilon e_i : \text{rk}(i) > 0\})$ of the claimed dimension. For base polytopes there are more interesting low-dimensional examples. The matroid case of the next proposition is [6, Prop 2.4].

Proposition 3.1. *The dimension of $P_{\text{base}}(\text{rk})$ is $|E|$ minus the largest number of parts in a partition $E = E_1 \dot\cup \cdots \dot\cup E_k$ such that $\text{rk}(E) = \text{rk}(E_1) + \cdots + \text{rk}(E_k)$.*

These sets E_i are known as the *connected components* of the polymatroid rk. In the case of matroids, the connected components are classically defined as the equivalence classes of the equivalence relation given by $i \sim j$ iff i and j appear together in some circuit. A (poly)matroid is called *connected* if it has only one connected component. In matroid theory there is distinctive terminology for connected components of one element i: if $\text{rk}(i) = 0$ then i is called a *loop*, while if $\text{rk}(i) = 1$ then i is called a *coloop*. We will speak of loops of polymatroids as well, though the notion of a coloop is less useful there.

Proof. Given any such partition, we have

$$\text{rk}(E) = \sum_{i \in E} x_i = \sum_{\ell=1}^{k} \sum_{i \in E_\ell} x_i \le \sum_{i=1}^{\ell} \text{rk}(E_i) = \text{rk}(E)$$

so each of the inequalities summed $\sum_{i \in E_i} x_i$ must be equality, implying that $P = P_{\text{base}}(\text{rk})$ is contained in k tranverse hyperplanes.

Conversely, for each part E_t of the partition, let S be a subset of E_t, and $a_1, a_2, \dots, a_m$ be the elements of $E_s \setminus S$. Write $A_i = \{a_1, \dots, a_i\}$. Not all of the equalities

$$\text{rk}(S \cup A_{i-1}) + \text{rk}(A_i) = \text{rk}(A_{i-1}) + \text{rk}(S \cup A_i)$$

can hold, or else their sum for $i = 1, \dots, m$ is also an equality, but cancelling terms appearing on both sides makes this

$$\text{rk}(S) + \text{rk}(E_t \setminus S) = 0 + \text{rk}(E_t)$$

which gives a partition with more parts than $\{E_i\}$. Similarly, if $S = \{s_1, \dots, s_\ell\}$, and $S_j = \{s_1, \dots, s_j\}$, not all of the equalities

$$\text{rk}(S_j \cup A_{i-1}) + \text{rk}(S_{j-1} \cup A_i) = \text{rk}(S_{j-1} \cup A_{i-1}) + \text{rk}(S_j \cup A_i)$$

may hold. Now Theorem 1.1 on the sequences

$$s_1, \dots, s_{j-1}, a_1, \dots, a_{i-1},\ s_j, a_i,\ \text{(other elements of } E\text{)}$$

and the same sequence with s_j and a_i exchanges gives two vertices of P differing by a multiple of $e_{s_j} - e_{a_i}$.

We use the previous argument iteratively. First let S be the set S_1 consist of a single element of E_t; this produces an element a_i, which we'll name b_1. Then let S be the set $S_2 = S_1 \cup \{b_1\}$; this produces another element a_i that we'll call b_2. Let $S_3 = S_2 \cup \{b_2\}$ and continue in this way until the sets S fill up all of E_t. At this point we have constructed $|E_t| - 1$ different segments in linearly independent directions $e_{s_j} - e_{a_i}$. Finally, summing over all parts E_t of the original partition, we have constructed segments in

$$\sum_{i=1}^{k} |E_i| - 1 = |E| - k$$

linearly independent directions in P, showing it is of dimension $|E| - k$. □

We can explicitly describe the face of either sort of polytope selected by any linear functional, in terms of minors. Every face of a base polytope is the base polytope of a minor; for independence polytopes, sums of base and independence polytopes can appear. We will use a few standard structural matroid notions. Given a polymatroid rank function rk and a subset S of its ground set E, the *deletion* and the *contraction* of S have ground set $E \setminus S$. The rank function $\mathrm{rk} \setminus i$ of the deletion is given by

$$(\mathrm{rk} \setminus S)(A) = \mathrm{rk}(A)$$

and that rk / i of the contraction by

$$(\mathrm{rk} / S)(A) = \mathrm{rk}(A \cup S) - \mathrm{rk}(S).$$

A *minor* of rk is a deletion of a contraction of rk. Deletions and contractions commute; we could equally have said a minor is a contraction of a deletion.

Proposition 3.2. *Let $f = \sum_{i \in E} f_i x_i$ be a linear functional. The face of $P_{\mathrm{base}}(\mathrm{rk})$ on which f is maximised is*

$$P_{\mathrm{base}}(\mathrm{rk} \setminus E_2 \cup \cdots \cup E_m) + P_{\mathrm{base}}(\mathrm{rk} / E_1 \setminus E_3 \cup \cdots \cup E_m) + \cdots \\ \cdots + P_{\mathrm{base}}(\mathrm{rk} / E_1 \cup \cdots \cup E_{m-1})$$

where the partition $E_1, \dots, E_m$ of E is such that $f_i > f_j$ if and only if $f_i \in E_k$ and $f_j \in E_\ell$ with $k < \ell$. (So each E_i consists of ground set elements with equal weight.)

The face of $P_{\mathrm{ind}}(\mathrm{rk})$ on which f is maximised is

$$P_{\mathrm{base}}(\mathrm{rk} \setminus E_2 \cup \cdots \cup F) + P_{\mathrm{base}}(\mathrm{rk} / E_1 \setminus E_3 \cup \cdots \cup F) + \cdots \\ \cdots + P_{\mathrm{base}}(\mathrm{rk} / E_1 \cup \cdots \cup E_{\ell-2} \setminus E_\ell \cup F) + P_{\mathrm{ind}}(\mathrm{rk} / E_1 \cup \cdots \cup E_{\ell-1} \setminus F)$$

where the partition $E_1, \dots, E_\ell, F$ of E is such that F contains all the elements i with $f_i < 0$, E_ℓ contains all the elements i with $F_i = 0$ (of which there may be none), and the remaining elements are partitioned as before.

The former polytope is a single base polytope, as we have claimed, because it is the base polytope of the *direct sum* of the individual polymatroids. Given a collection of polymatroids rk_i on disjoint ground sets E_i, their direct sum has ground set $\dot{\bigcup} E_i$ and rank function $rk(A) = \sum_i \mathrm{rk}_i(A \cap E_i)$.

A sort of converse holds:

Corollary 3.3. *Let* rk *be a polymatroid and* $\mathrm{rk}' = \mathrm{rk} \setminus S/T$ *any minor thereof. Then one of the faces of* $P_{\mathrm{base}}(\mathrm{rk})$ *is the translate of* $P_{\mathrm{base}}(\mathrm{rk}')$ *by* v*, where* v *is any vertex of* $P_{\mathrm{base}}(\mathrm{rk} \setminus (E \setminus T))$*.*

To obtain this face choose the functional f to give distinct large positive weights to T that select the vertex v from $P_{\mathrm{base}}(\mathrm{rk} \setminus (E \setminus T))$, distinct negative weights to S, and weight zero to everything else.

Proof. We can prove this by analysis of the greedy algorithm. Since we have defined the greedy algorithm for independence polytopes, we give the proof in that context. The base polytope version can be reduced to this straightaway. The sum of the coordinates being constant on $P_{\mathrm{base}}(\mathrm{rk})$, we can add any multiple of it to f without changing the face selected. In particular we can make all coefficients of f positive, and then f selects the same face of the base polytope as of the independence polytope.

The vertices of the face of $P_{\mathrm{ind}}(\mathrm{rk})$ where f is maximised are the vertices on which $f + \epsilon$ are maximised, as ϵ varies over sufficiently small vectors. If all coefficients of $f+\epsilon$ are distinct and none of them is zero, then the greedy algorithm has no choices to make, optimising x_i in turn for the positive f_i in decreasing order and leaving the other x_i zero. Our choice of ϵ can only make a difference in this process in the event that some of the coefficients of f were zero or equal to start with. Thus, given that our E_1 consists of all i such that f_i is maximal and positive, the greedy algorithm begins by selecting a vertex of $P_{\mathrm{base}}(\mathrm{rk} \setminus (E \setminus E_1))$, and the various choices of ϵ allow selection all of its vertices. It next moves on to E_2, consisting of the i where f_i is next largest, and maximises these subject to the choices made for E_1. This selects a vertex of $P_{\mathrm{base}}(\mathrm{rk}/E_1 \setminus (E \setminus (E_1 \cup E_2)))$. We continue in this fashion until reaching the negative coefficients of f, where we stop. If any coefficients of f are zero, those in the set we have called E_ℓ, then our choice of ϵ allows some to be included and others excluded in the greedy process, and in this way we can achieve every vertex of $P_{\mathrm{ind}}(\mathrm{rk}/E_1 \cup \cdots \cup E_{\ell-1} \setminus F)$. The choices made in ϵ here are independent, pertaining to different coefficients, so overall the vertices attainable are exactly the vertices of the Minkowski sum of the polytopes for each step. This is the claim of the proposition. □

Which of the inequalities of (poly)matroid polytopes define facets, i.e. are nonredundant? Separate answers have appeared in the literature for the independence polytope and the base polytope; we state both here.

Proposition 3.4 (Feichtner-Sturmfels [6, Prop 2.6])**.** *The inequality for the set* A *defines a facet of* $P_{\mathrm{base}}(\mathrm{rk})$ *if and only if* $\mathrm{rk} \setminus (E \setminus A)$ *and* rk/A *are both connected.*

Proposition 3.5 (Edmonds)**.** *The inequality for the set* A *defines a facet of* $P_{\mathrm{ind}}(\mathrm{rk})$ *if and only if* $\mathrm{rk} \setminus (E \setminus A)$ *is connected and* rk/A *contains no loops.*

Sets A such that rk/A contain no loops are generally known as *flats*; they are maximal sets of their rank. We prove both propositions together.

Proof. The face F of either sort of polytope P where the inequality for the set A is tight is the face maximising the functional $\sum_{i\in A} x_i$. Proposition 3.2 describes F as the Minkowski sum of two polytopes in disjoint sets of coordinates, those indexed by $E_1 = A$ and those indexed by $E_2 = E \setminus A$, so the dimension of this sum is the sum of the dimensions of the summands. The first summand is a base polytope, satisfying the extra equality $\sum_{i\in A} x_i = \operatorname{rk}(A)$ that P itself does not satisfy. So F is a facet if and only if there are no further equalities, i.e. both the summand for $\operatorname{rk}\setminus(E \setminus A)$ and that for rk/A have the largest possible dimension. But this follows from Proposition 3.1 and the foregoing discussion. □

Finally, we have defined (poly)matroid polytopes in terms of permissible directions of their defining inequalities. At the opposite end of the dimension gamut, there is an equally pretty characterisation of these polytopes in terms of their vertices and edges. The matroid base polytope version came first, and is due to Gelfand, Goresky, MacPherson and Serganova [8].

Theorem 3.6. *A polytope is a matroid, respectively a polymatroid, base polytope if and only if all coordinates of each of its vertices are 0 or 1, respectively positive, and each of its edges is in a direction of form $e_i - e_j$ for some $i, j \in [n]$.*

A polytope is a matroid, respectively a polymatroid, independence polytope if and only if all coordinates of each of its vertices are 0 or 1, respectively positive, the origin is one of these vertices, and each of its edges is in a direction of form e_i or $e_i - e_j$ for some $i, j \in [n]$.

Proof. We only do the case of base polytopes in full here. We already know the characterisation of the vertices. The complete description of the faces in Proposition 3.2 tells us that any edge must be a polymatroid base polytope itself. Now if rk is a polymatroid whose base polytope is an edge, then it cannot contain three elements of positive rank, for if so its independence polytope contains a small 3-simplex i.e. has dimension at least 3, and the base polytope, which is a facet thereof, has dimension at least 2. Removing loops, we may assume that the ground set of rk has size at most 2. So its base polytope is contained in the 2-space spanned by e_i and e_j for some i and j, and has constant coordinate sum; thus it must be an edge in direction $e_i - e_j$.

For the converse, suppose P is a polytope all of whose edges are in directions $e_i - e_j$. Let F be a face of P. The space of linear forms $\sum c_i x_i$ which are constant on F equals the intersection of the corresponding spaces for its edges, and each of these is a hyperplane $\{c_i = c_j\}$, so the corresponding space for F is of the form

$$\{(c_i) \in \mathbb{R}^n : c_i = c_j \text{ if } i \sim j\}$$

for some equivalence relation $\sim$ of $[n]$, the transitive closure of the relation with $i \sim j$ iff an edge in direction $e_i - e_j$ appears. Therefore F is the set of solutions to a system of inequalities made up of equalities

$$\sum_{i\in\Pi} x_i = b_\Pi \tag{3.1}$$

for each equivalence class Π of $\sim$, as well as a single defining inequality for each facet. Since the facets also have descriptions of the same form, the defining inequalities can also be taken to be of the form

$$\sum_{i\in J} x_i < b_J$$

for some $J \subseteq E$. It follows that there exists some function rk, not necessarily submodular, so that $P = P_{\text{base}}(\text{rk})$.

It remains to show that rk may be taken submodular. The particular choice we make is

$$\text{rk}(J) = \max_{x\in P} \sum_{i\in J} x_i.$$

For a subset $J \subseteq [n]$, let $x_J = \sum_{i\in J} x_i$. Let A and B be any two subsets of $[n]$. Let F be the face of P on which $x_{A\cap B}$ is maximised, and G the face of F on which $x_{A\cup B}$ is maximised. For sufficiently small $\epsilon > 0$, we can obtain the same face G by first maximising $x_{A\cap B} + \epsilon x_{A\cup B}$, giving a face H, and then maximising $x_{A\cup B}$. But the equalities satisfied by H have a basis of the form (3.1), so given that $x_{A\cap B} + \epsilon x_{A\cup B} = c$ is among them for some c, so must also be $x_{A\cap B} = c'$ and $x_{A\cup B} = c''$. In fact, c' and c'' must be entirely independent of ϵ as long as it remains positive, since otherwise there would be some critical value of ϵ picking out a face where $x_{A\cap B}$ or $x_{A\cup B}$ was not constant, which the same argument rules out. Therefore, using a very large $\epsilon \gg 0$, if we first maximise $x_{A\cup B}$ and then $x_{A\cap B}$, we pick out the very same face G. We conclude that the value of $x_{A\cap B} + \epsilon x_{A\cup B}$ on G equals $\text{rk}(A\cap B) + \text{rk}(A\cup B)$. But $x_{A\cap B} + \epsilon x_{A\cup B} = x_A + x_B$, so this feasible value provides a lower bound for $\text{rk}(A) + \text{rk}(B)$. Therefore rk is submodular.

It remains just to deal with the conditions on the vertices. But if P is in the positive orthant then clearly rk takes positive values, so is a polymatroid rank function; and if all vertices of P are zero-one vectors, then all the optima defining rk are attained at one of these zero-one vectors, so that $\text{rk}(J)$ is integral and at most $|J|$ for each J, making rk a matroid rank function.

Finally, we speak in brief to independence polytopes. The proof can basically be adapted *mutatis mutandis*. The key for the converse direction is, notionally, to add a slack variable x_0 for the inequality $\sum_{i\in E} x_i \le \text{rk}(E)$ of the independence polytope. This corresponds to passing from the putative independence polytope P to a polytope Q in the hyperplane $x_0 + \sum_{i\in E} x_i = c$ projecting to it. Translating the edge direction conditions to Q, the allowable directions become exactly the directions $e_i - e_j$. So Q is a polymatroid base polytope, which projects to an independence polytope on dropping the slack variable. □

Here is a bonus proof for the matroid base polytope case, using the basis exchange formulation of matroids.

Bonus proof of 3.6 for matroid base polytopes. Given an edge $\mathcal{E}$ of a matroid base polytope P, let f be a linear functional minimised exactly on $\mathcal{E}$, and let B_0 and B_1 be the bases corresponding to its ertices. Suppose, without loss of generality, that an element j of maximal f-weight in their symmetric difference lies in B_0. By the base exchange axiom, there is a basis $B_0 \setminus \{j\} \cup \{i\}$ for some $i \in B_1 \setminus B_0$, but i has weight no greater than j, so this new basis must lie on $\mathcal{E}$ and must be B_1. Hence $\mathcal{E}$ is in direction $e_i - e_j$.

Conversely, if P is a polytope with the given conditions on vertices and edges, we must prove the base exchange axiom for the collection of bases

$$\{B : \sum_{i\in B} e_i \text{ is a vertex of } P\}.$$

Let B_0 and B_1 be two bases in this set and $j \in B_0 \setminus B_1$, and consider the linear functional $f = x_j + \sum_{k\in[n]\setminus(B_0\cup B_1)} x_k$. This takes value 0 at B_1 so is not minimised at B_0, where its value is 1. By elementary linear programming, there must be an edge leaving B_0 along which the value of f decreases; since the basis at the other end of the edge has form $B_0 \setminus \{k\} \cup \{\ell\}$ by assumption this is only possible if $k = j$ and $\ell \in B_0 \cup B_1 \setminus j$, indeed $\ell \in B_1 \setminus B_0$ since a basis cannot contain an element multiply. This proves the exchange condition. □

4. Subdivisions

A *polyhedral cell complex* S is a finite collection of polyhedra called *cells*, closed under taking faces, and such that the intersection of two polyhedra in S is a face of each, the empty set being allowed as a face. Such a cell complex is a *subdivision* of a polyhedron P when P is the union of its members. The *trivial* subdivision of a polyhedron P consists of the set of faces of P itself.

Example 4.1. Here are a couple sorts of behaviours disallowed in regular subdivisions, both of which can be observed in one dimension.

- The set
$$\{[0,2],[1,2],[1,3],\{0\},\{1\},\{2\},\{3\}\}$$
is not a subdivision of the interval $[0,3]$ because its cells $[0,2]$ and $[1,3]$ intersect in $[1,2]$, which despite being present in the collection is not a face of the intervals intersected.
- The set
$$\{\{x\} : x \in [0,1]\}$$
is not a subdivision of the interval $[0,1]$ because of the finiteness condition.

Here are a few basic standard facts about subdivisions.

Proposition 4.2. *The relative interiors of the faces of any subdivision of P partition P.*

Proof. Let S be a subdivision of P. Every point $x \in P$ lies in some polyhedron Q in S, and if F is the intersection of all faces of Q containing x, then x is in F but no face thereof, i.e. in the relative interior of F. Now if x were in the relative interior of another polyhedron G in S, then the intersection $F \cap G$ would contain a point in the relative interior of each; since this intersection must be a face of each of the two, we must have $F = F \cap G = G$. □

Proposition 4.3. *A subdivision of P is determined by its maximal-dimensional cells, all of which have dimension equal to* $\dim(P)$*.*

See [3, Section 2.3] for a proof, minding differences between their setup and ours.

Example 4.4. The easiest nontrivial subdivisions are those with only two maximal-dimensional cells. These are known as the *hyperplane splits*. They are all obtained by letting H be a hyperplane dividing the ambient space into halfspaces H_1 and H_2 such that P intersects the interiors of both halfspaces, and taking $P \cap H_1$ and $P \cap H_2$ as the maximal-dimensional cells of the subdivision.

The *indicator function* $\mathbb{1}_P$ of a polytope P in a vector space V is the function $f : V \to \mathbb{Z}$ taking value 1 on P and 0 elsewhere.

Proposition 4.5. *Let S be a subdivision of a polyhedron P, and $S_1, \ldots, S_k$ its faces of maximal dimension. Then*

$$\mathbb{1}_P = \sum_{\emptyset \neq A \subseteq [k]} (-1)^{|A|-1} \cdot \mathbb{1}_{\bigcap_{i \in A} S_i}.$$

4.1. **Matroid subdivisions.** A *(poly)matroid subvidision* is a subdivision of a (poly)matroid base polytope whose maximal cells, and therefore all of whose cells, are also (poly)matroid base polytopes. We focus on the matroid case in these notes.

Beyond the strand of work in linear programming and combinatorial optimisation that has been the focus of Edmonds' part of this course, matroid base polytopes have appeared in several other contexts. One of these associates matroids to the type A Coxeter groups, observing that their edge directions are the roots of the type A root systems, and generalises to other Coxeter groups; these are the subject of the book [1], though the subject still seems to be awaiting its killer application.

Algebraic geometry has seen many uses of the matroid base polytope, including the construction of the matroid stratification of the Grassmannian [8]. A number of its other uses ultimately trace back to the fact that matroid polytope subdivisions encode toric degenerations of certain toric varieties: these include the moduli space of hyperplane arrangements (Hacking, Keel and Tevelev [9] and Kapranov [10]), compactifying fine Schubert cells in the Grassmannian (Lafforgue [12, 11]), and tropical linear spaces (Speyer [16]). This work has many close connections to questions of realisability, as Lafforgue's work indicates (our Theorem 7.8).

From the enumerative or invariant-theoretic point of view, many invariants of interest, like the Tutte polynomial, satisfy relations arising from subdivisions, so knowing that one's matroid is subdivided may help prove assertions about these invariants.

One simple reason we don't concern ourselves with independence polytope subdivisions is because there aren't any nontrivial ones, even for polymatroids. To see this, observe that the independence polytope of any polymatroid rk contains the convex hull of the origin and the vectors $\mathrm{rk}(\{i\})e_i$ for each i. If $\mathrm{rk}(i) = 0$ for some i, then the independent set polytope P of rk lies in the hyperplane $x_i = 0$ and is not full-dimensional. But if all $\mathrm{rk}(\{i\})$ are positive then P contains a small dilate of the simplex $\mathrm{conv}\{0, e_1, \ldots, e_n\}$, and only one top-dimensional polytope in a subdivision may do so by Proposition 4.2.

Aside. In the light of Proposition 4.5 the difference between the two kinds of polytopes is, in fact, unimportant. Given a collection of matroids M_i, one has $\sum_i c_i \mathbb{1}_{P_{\mathrm{base}}(M_i)} = 0$ if and only if $\sum_i c_i \mathbb{1}_{P_{\mathrm{ind}}(M_i)} = 0$. This is because the passage from base to indicator polytopes in Proposition 1.3 is linear on indicator functions. The passage in the other direction is also linear when restricted to polytopes whose maximal coordinate sum is the same, and the linear relations among indicator polytopes are generated by relations among those whose maximal coordinate sum is the same.

4.2. **Matroids without subdivisions.** One of the first questions one might have about a matroid is whether its polytope permits any nontrivial subdivisions at all. This predicate was one half of one of the first interesting theorems on the subject, Theorem 7.8. In some cases the answer is "no". We present a few examples here.

The *beta invariant* of a matroid M with rank function rk on ground set E is defined as

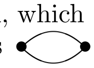
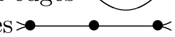
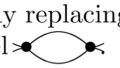
$$\beta(\mathrm{rk}) = \sum_{A \subseteq E} (-1)^{\mathrm{rk}(E)-|A|}\,\mathrm{rk}(A).$$

(It appears as the coefficient of x and of y in the Tutte polynomial.)

Proposition 4.6. *If S is a matroid subdivision with maximal-dimensional faces $S_1, \dots, S_k$, and the corresponding matroids are M and $M_1, \dots, M_k$, then*

$$\beta(M) = \beta(M_1) + \cdots + \beta(M_k).$$

The beta invariant of a matroid is a nonnegative integer. It equals zero if and only if the matroid is disconnected (i.e. has base polytope of dimension less than $|E| - 1$), and one if and only if the matroid is series-parallel. *Thus, series-parallel matroids allow no nontrivial matroid subdivisions.*

A series-parallel matroid is the graphic matroid of a series-parallel graph, which in turn is a graph that can be built from the graph with two parallel edges  by repeatedly replacing an edge with two edges either in series  or in parallel .

We omit the proof, as it goes somewhat afield into enumerative combinatorics. But we do give an example of a non-subdivisible matroid which is not series-parallel.

Proposition 4.7 (Chatelain-Ramírez Alfonsin [2, Thm 3])**.** *Matroids realisable over $\mathbb{F}_2$ have no matroid subdivisions which are hyperplane splits.*

Proof. Let M be realisable over $\mathbb{F}_2$, and suppose its base polytope P had a hyperplane split into top-dimensional pieces P_1 and P_2. In view of the defining inequalities of the base polytope, the equation of the hyperplane must have the form $\sum_{i \in A} x_i = b$ for some set $A \subseteq E$ and integer b. There must then be a lattice point of P where $\sum_{i \in A} x_i$ attains the value $b - 1$, and another where it attains the value $b + 1$; otherwise either the minimum or maximum of this functional is b, meaning that P doesn't intersect one of the open halfspaces it's supposed to and one of P_1 and P_2 has too low dimension.

Let B_1 and B_2 be bases of M corresponding to such lattice points, so $|B_1 \cap A| = b-1$ and $|B_2 \cap A| = b+1$. Use the basis exchange axiom to replace each element of $(B_1 \cap A) \setminus B_2$ with an element of B_2. The resulting basis B_1' has $B_1' \cap A \subseteq B_2 \cap A$ and $|B_1 \cap A| \le b-1$; use further basis exchange to replace elements of B_1' with those of B_2 until obtaining a basis B_1'' with $|B_1'' \cap A| = b - 1$ again and $B_1'' \cap A \subseteq B_2 \cap A$ still. Now repeat the process with the roles of B_1 and B_2 reversed. The final result are two bases B_1'' and B_2'' of M that can be written $B_1'' = S \,\dot\cup\, \{a, b\}$ and $B_2'' = S \,\dot\cup\, \{c, d\}$ with $|S \cap A| = b - 1$, $a, b \notin A$, and $c, d \in A$.

Consider the minor $M' = M/S \setminus (E \setminus S \setminus \{a,b,c,d\})$. This is a matroid of rank 2 on the ground set $\{a,b,c,d\}$, of which $\{a,b\}$ and $\{c,d\}$ are bases. Its base polytope, which is a face of P per Corollary 3.3, therefore has a hyperplane split by the hyperplane $x_a + x_b = 1$, which is a matroid subdivision because its top-dimensional cells are faces of P_1 and P_2 respectively. By a theorem of Tutte, since M is realisable over $\mathbb{F}_2$, M' cannot be isomorphic to the uniform matroid $U_{2,4}$. But there is no other base polytope with $e_{\{a,b\}}$ and $e_{\{c,d\}}$ as vertices on which this hyperplane split induces a matroid subdivision. □

Example 4.8. The graphic matroid of the complete graph K_4, like any graphic matroid, is binary. One computes that its beta invariant equals 2. By Proposition 4.7 it has no hyperplane split, i.e. subdivision into two top-dimensional pieces, while by Proposition 4.6 it has no subdivision into more than two top-dimensional pieces. So it has no nontrivial subdivisions of any kind.

5. Regular subdivisions

Regular subdivisions are, in my opinion, the most tractable kind of subdivision. We will look at examples of regular and non-regular matroid subdivisions later.

Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection which simply discards the last coordinate. Informally, it will be linguistically useful to speak of this last coordinate as "height", so that e.g. increasing its value is moving "up".

Let $P \subseteq \mathbb{R}^n$ be any polyhedron, and Q another polyhedron such that $\pi(Q) = P$ and such that Q contains no ray in the direction $-e_{n+1}$. Then the *regular subdivision* $S(Q)$ *induced by* Q is the collection of all faces of the polyhedra $\pi(F)$, where F is a facet of Q such that an affine-linear functional c which is zero on F, and nonnegative but not everywhere zero on Q, has a positive coefficient of x_{n+1}.

Aside. We sometimes regard the non-constant coefficients of such a functional

$$c = c_0 + c_1x_1 + \cdots + c_{n+1}x_{n+1}$$

as giving a *normal vector* $(c_1, \ldots, c_{n+1})$ in the dual $(\mathbb{R}^{n+1})^*$ to the facet F. This normal vector is *inner* if c takes nonnegative values on Q, as we have required; if c instead took nonpositive values on Q it would give an *outer* normal vector.

Here's a restatement of the definition using the inequality description of Q. Consider a finite collection $f_1, \ldots, f_k$ of affine-linear functionals. For each i, let G_i be the set of points of P at which f_i attains the largest value among any of these functionals (allowing ties). Then the set of all faces of the polyhedra G_i is the regular subdivision $S(Q)$, where

$$Q = \{(x,h) : x \in P, h(x) \geq f_i(x) \text{ for all } i\},$$

and any regular subdivision can be restated in this form.

All elements of $S(Q)$ are projections of faces of Q. The faces which appear are those which minimise a collection of linear functionals f_i at least one of which has positive coefficient of x_{n+1}. Any convex combination $\sum c_i f_i$ with the c_i strictly positive is minimised on the same face, and the convex combination can be chosen to leave the coefficient of x_{n+1} positive. To restate that: if F is a face of Q, then $\pi(F)$ appears in S if and only if translates $F - \epsilon e_{n+1}$ of this face downward do not intersect Q, for small $\epsilon > 0$. Let us call such faces F of Q *lower*.

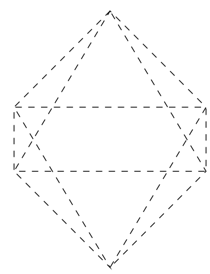


FIGURE 1. The octahedron of Example 5.3.

Proposition 5.1. *The regular subdivision $S(Q)$ is a subdivision of P.*

Proof. If $\pi(F)$ is a polytope in $S(Q)$, the image of the lower face F of Q, and G is a face of $\pi(F)$ where the functional f is minimised, then $f \circ \pi$ is minimised on a face $\tilde{G}$ of F projecting to G. Since F is lower, so is $\tilde{G}$, so G is in $S(Q)$. Next, let $\pi(F)$ and $\pi(G)$ be polytopes in $S(Q)$ and x an intersection point. The only preimage of x under π which can be contained in a lower face is the element of $\pi^{-1}(x) \cap Q$ with minimal last coordinate. So in fact $F \cap G$ contains this preimage. We conclude that $\pi(F) \cap \pi(G) = \pi(F \cap G)$, which is in $S(Q)$. Finally, $\pi(Q) = P$ so Q subdivides P. □

The following is an immediate consequence of Theorem 3.6.

Corollary 5.2. *The subdivision $S(Q)$ is a matroid subdivision if any only if, for every lower vertex v of Q, all coordinates of $\pi(v)$ lie in $\{0,1\}$, and for every lower edge $\mathcal{E}$ of Q, the edge $\pi(\mathcal{E}) \subseteq \mathbb{R}^n$ is in the direction $e_i - e_j$ for some $i, j \in [n]$.*

The same is true for polymatroid subdivisions when $\{0,1\}$ is replaced by $\mathbb{R}_{\geq 0}$.

Example 5.3. The smallest matroid base polytope with a nontrivial subdivision is the octahedron $P = \operatorname{conv}\{e_{\{i,j\}} : 1 \leq i < j \leq 4\}$. associated to the uniform matroid $U_{2,4}$; see Figure 1. It possesses seven regular subdivisions. The three finest are all related by the action of the symmetric group S_4 on the ground set; One of them has maximal faces

$$\operatorname{conv}\{e_{12}, e_{13}, e_{14}, e_{34}\}, \qquad \operatorname{conv}\{e_{12}, e_{13}, e_{23}, e_{34}\}, \tag{5.1}$$

$$\operatorname{conv}\{e_{12}, e_{14}, e_{24}, e_{34}\}, \qquad \operatorname{conv}\{e_{12}, e_{23}, e_{24}, e_{34}\}, \tag{5.2}$$

e_{12} being short for $e_{\{1,2\}}$ and so on. These all share the "long" edge $\operatorname{conv}\{e_{12}, e_{34}\}$, so are not matroid base polytopes. The three next coarsest regular subdivisions, again all related by the S_4 action, are hyperplane splits, a representative one having maximal faces

$$\operatorname{conv}\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}\}, \qquad \operatorname{conv}\{e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}; \tag{5.3}$$

see Figure 2. These are matroid subdivisions. The last regular subdivision, of course, is the trivial one.

Which of these subdivisions is attained depends on which of the three quantities

$$p_{12} + p_{34}, \quad p_{13} + p_{24}, \quad p_{14} + p_{23}$$

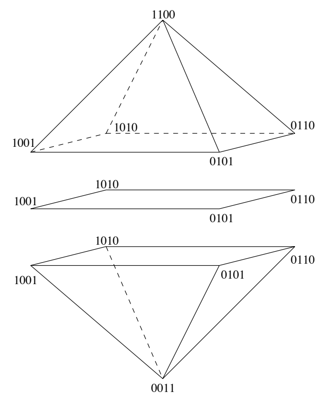


Figure 2. The nontrivial matroidal subdivision of Example 5.3, depicted as its two maximal faces and their intersection.

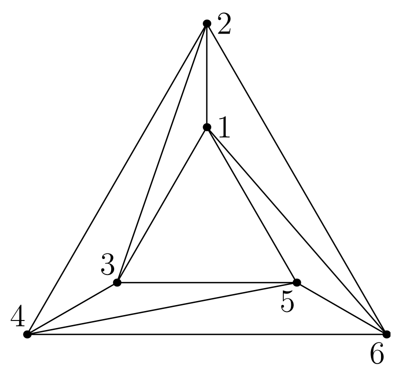


Figure 3. A nonregular subdivision.

is least, where p_J is the height to which vertex e_J is lifted, i.e. $Q = \mathrm{conv}\{(e_J, p_J)\}$ is the determining polytope. If a unique one of these three sums is least, it determines a long edge through the middle of the octahedron; for example when $p_{12} + p_{34}$ is least we get subdivision (5.1). When two are tied for least, they determine a square through the middle: thus in (5.3) we have $p_{13}+p_{24} = p_{14}+p_{23} < p_{12}+p_{34}$. Finally, when all three are equal, the height function is in fact an affine linear function on the whole octahedron and we get a trivial subdivision.

We will call on this example in Section 7.2.

It seems remiss not to state a general theorem which sees expression in this example, regarding regular subdivisions which do not introduce new vertices.

Theorem 5.4. *[[secondary polytope]]*

We omit the proof.

Example 5.5. The standard example of a nonregular subdivision, sometimes called "the mother of all examples", is depicted in Figure 3. Let $x_1, \dots, x_6$ be its six points, according to the labelling in the figure. If this is to arise as $S(Q)$ then Q must have just six lower vertices, projecting to the x_i; call them (x_i, h_i).

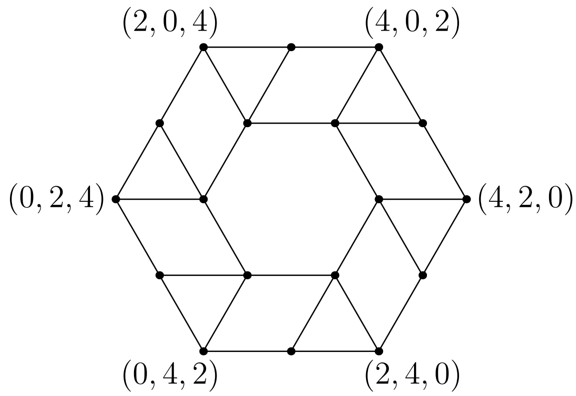


Figure 4. A nonregular subdivision of polymatroid base polytopes.

Let $w = ax_3 + (1-a)x_2 = ax_1 + (1-a)x_4$ be the point of intersection: a reappears on both sides since x_1x_3 and x_2x_4 are parallel. From the figure, $\operatorname{conv}\{(x_2, h_2), (x_3, h_3)\}$ must be a lower edge of Q. On the other hand the segment $\operatorname{conv}\{(x_1, h_1), (x_4, h_4)\}$ is not contained in a lower face of Q. So the point $(w, ah_3+(1-a)h_2)$ can be translated by a sufficiently small multiple of $-e_{n+1}$ while remaining in Q, whereas this is false for the point $(w, ah_1 + (1-a)h_4)$. These two midpoints differ only in the last coordinate, so $ah_3 + (1-a)h_2 > ah_1 + (1-a)h_4$, i.e.

$$ah_1 - (1-a)h_2 < ah_3 - (1-a)h_4.$$

By symmetries of the figure we also have

$$ah_3 - (1-a)h_4 < ah_5 - (1-a)h_6, ah_5 - (1-a)h_6 < ah_1 - (1-a)h_2,$$

giving a contradiction.

Example 5.6 (Speyer)**.** Figure 4 is nonregular for very much the same reason as in the last example. It is also a subdivision of integer polymatroids, if the plane of the picture is taken to be the plane $x_1+x_2+x_3 = 6$, with coordinates as suggested in the figure, by Corollary 5.2.

This can be expanded to a nonregular matroid subdivision. Let $\rho : \mathbb{R}^{12} \to \mathbb{R}^3$ be the projection

$$\rho(x_1, \dots, x_{12}) = (x_1 + \cdots + x_4, x_5 + \cdots + x_8, x_9 + \cdots + x_{12}),$$

and let S be the subdivision consisting of the polytopes $\rho^{-1}(P) \cap [0, 1]^{12}$ for each polytope P in the figure, and their faces. This subdivision cannot be a regular subdivision $S(Q)$, or else the figure would also be a regular subdivision, arising as $S(\tilde\rho(Q))$ where $\tilde\rho((x, h)) = (\rho(x), h)$.

To see that S is a matroid subdivision, let $P = P_{\text{base}}(\text{rk})$ be one of the cells of the subdivision of the figure. Define $\bar\rho : 2^{[12]} \to 2^{[3]}$ so that $\bar\rho(A)$ is the support of $\rho(e_A)$. Then the set function $\widetilde{\text{rk}} = \text{rk} \circ \bar\rho$ is still submodular. If $x \in P_{\text{base}}(\widetilde{\text{rk}})$ then $\rho(x) \in P$, by considering the defining inequalities for unions of parts of the relevant partition $\Pi = \{1, \dots, 4\}, \{5, \dots, 8\}, \{9, \dots, 12\}$. Conversely, if x is a vector with coordinates in the interval $[0, 1]$ and with $\rho(x) \in P$, then x satisfies the inequalities of $P_{\text{base}}(\widetilde{\text{rk}})$ coming from unions of parts of Π, and the others are weaker by positivity of the coordinates. So the polytope in S corresponding to P is $P_{\mathrm{base}}(\widetilde{\mathrm{rk}}) \cap [0,1]^{12}$, and this is a matroid base polytope by the proof of the matroid union theorem.

6. A fertile example

This example is fertile because we'll be able to regard several others as springing from it. The example is a theme of [7], though this is far from its only appearance.

Let $A = (a_{ij})$ be an $r \times n$ matrix with entries in $\mathbb{R} \cup \{\infty\}$. For each subset $J \subset [n]$ of size r, define p_J to be the minimum of $\sum_{i \in [r]} a_{i,\sigma(i)}$ as σ ranges over bijections from $[r]$ to J. In other words, p_J is the solution to the (minimum) assignment problem for the maximal square submatrix $A_{[r],J}$. We assume that not all the p_J equal ∞ (in particular that $r \le n$).

Proposition 6.1. *The regular subdivision induced by the polytope*

$$Q_A := \mathrm{conv}\{(e_J, p_J) : p_J \neq \infty\}$$

is a matroid subdivision.

We will prove this in the course of unravelling and generalising the construction.

Example 6.2. We give a small example, though not so small as to be contentless: let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Then

$$Q_A = \mathrm{conv}\{(e_{12}, 0), (e_{12}, 0), (e_{12}, 0), (e_{12}, 0), (e_{12}, 0), (e_{12}, 0)\}$$

and the induced subdivision of the octahedron $P_{\mathrm{base}}(U_{2,4})$ is the one portrayed in Figure 2, whose cells are matroid polytopes.

6.1. **Matroid unions.** The first question is which matroid base polytope $S(Q_A)$ gives a subdivision of. The answer is

$$\pi(Q_A) = \mathrm{conv}\{e_J : p_J \neq \infty\}.$$

The minimum p_J avoids taking the value ∞ so long as there is at least one bijection $\sigma : [r] \to J$ such that all the $a_{i,\sigma(i)}$ are finite. To simplify the data, define an auxiliary bipartite graph G_A on bipartitioned vertex set $[r] \amalg [n]$ such that (i,j) is an edge of G_A if and only if $a_{i,\sigma(i)} < \infty$. Then the condition that the permutation σ contribute a term making p_J finite is precisely the condition that

$$\{(i, \sigma(i)) : i \in [r]\}$$

be a maximal matching in $G(A)$. It follows that what we have in $\pi(Q_A)$ is the base polytope of a *transversal matroid*:

Definition 6.3. Let G be a bipartite graph on bipartitioned vertex set $V \amalg W$. The independent sets of the *transversal matroid* of G are the subsets $\{v_1, \ldots, v_k\}$ of V such that there exist edges $e_1, \ldots, e_k$ sharing no vertices in common and such that v_i is a vertex of e_i for each i. Such sets of edges are known as *matchings* or *transversals.*

We must prove that a transversal matroid is a matroid. This is in fact immediate: transversal matroids are exactly the unions of matroids of rank 1. Indeed, given a graph G as above, define a matroid M_w for each $w \in W$ of rank 1, whose bases are the singletons $\{v\}$ such that (v, w) is an edge of G (and whose only other independent set is empty). Then the union of the M_w is the transversal matroid of G, by the matroid partitioning theorem. Conversely, given a family of matroids of rank 1 it is clear how to invert this process and construct G.

It is pleasant that Hall's renowned marriage theorem comes as a direct corollary of this approach, via the submodular-function definition of the matroid union that characterises its dependent sets.

Theorem 6.4 (Hall)**.** *Let G be a bipartite graph on bipartitioned vertex set $V \amalg W$. There exists a transversal meeting V in the subset J if and only if there exists no subset $K \subseteq J$ with fewer than $|K|$ neighbours in W.*

Proposition 6.5. *All transversal matroids M are linearly realisable.*

Proof. If $\mathbf{k}$ is any field, then M may be realised over a transcendental extension of $\mathbf{k}$ with a transcendental $x_{v,w}$ adjoined for each edge (v, w) of G: the matrix $A = (a_{v,w})$ gives a realisation where $a_{v,w} = 0$ if (v, w) is a nonedge of G, and $a_{v,w} = x_{(v,w)}$ is an edge of G. This gives a realisation because of the permutation expansion of the determinant. The minor of A on column set $J \subseteq W$ is nonzero if any of its permutation summands is nonzero, because these summands are distinct monomials, among which there can be no cancellation. But such a summand is exactly a matching from V to J. □

If $\mathbf{k}$ is sufficiently large (for instance, $\mathbf{k} = \mathbb{R}$) then this gives a realisation over $\mathbf{k}$ itself, by specialising the indeterminates to generic elements of $\mathbf{k}$: algebraically independent choices will always be generic enough.

Note that the realising matrix constructed in the last proof might have more rows than its rank. For the purposes that follow, such as to apply Corollary 2.5, we'd like to have a representation with only as many rows as the rank. But the existence of one is immediate from the linear algebra: among the vector space spanned by the set of rows, there must be a basis; restrict to these rows. The combinatorial translation of this statement doesn't look quite so trivial:

Corollary 6.6. *Let G be a graph on bipartite graph on bipartitioned vertex set $V \amalg W$. If its transversal matroid has rank r, then there is a subset J of V with size r such that the induced subgraph on $J \amalg W$ has the same transversal matroid.*

Proof of Proposition 6.1. Given such an $r \times n$ matrix A, for each $i = 1, \dots, r$, define a polytope

$$Q_i = \operatorname{conv}\{(e_j, a_{i,j}) : a_{i,j} \neq \infty\}.$$

Let $Q = \sum_{i=1}^{r} Q_i$. Every face of Q is a Minkowski sum of faces of the Q_i. In particular, every vertex of Q must be a sum of vertices of the Q_i; the projection of any such vertex under π is a sum of vectors of form e_j, so has positive integral coordinates. Every edge $\mathcal{E}$ of Q must be a sum of vertices or edges of the Q_i, and any edges that occur as summands must be parallel. All edges of the Q_i project under π to an edge of form (e_j, e_k) for some $j, k \in [n]$, so all the edges that are summands of $\mathcal{E}$ must share this projection and the projection of $\mathcal{E}$ itself must have the same direction vector, namely $e_j - e_k$. Therefore Q satisfies the conditions of Lemma 5.2, and induces a polymatroid subdivision.

From here, we see that

$$Q_{\mathrm{cube}} := Q \cap \pi^{-1}([0,1]^n)$$

induces a matroid subdivision. For any top-dimensional face of $S(Q_{\mathrm{cube}})$, let F be the lower facet of Q_{cube} which projects to it. Then the intersection of the affine span of F with Q is a lower facet $\hat{F}$ of Q, projecting to some maximal face in $S(Q)$, which is a polymatroid base polytope. Since the projection π commutes with intersecting with the unit cube (on the coordinates not discarded), $\pi(F)$ is a matroid base polytope. Indeed it is the base polytope of the transversal matroid of A, by Corollary 2.5 and the assumption that not all p_J equal ∞. In particular, we get that all lower vertices of Q_{cube} have integer coordinates, except possibly for the last.

What's more, matroid partitioning also implies that the lower vertices of Q_{cube} must be sums of vertices of the Q_i. Let v be a vertex of Q_{cube}. Let F be the face of Q on whose relative interior v lies. Then F is the Minkowski sum of faces F_i of Q_i, which project under π to base polytopes $P_{\mathrm{base}}(\mathrm{rk}_i)$ of rank one matroids. Therefore $\pi(v)$ lies in the unit cube and satisfies the inequalities

$$\sum_{i \in J} \pi(v)_i \leq \mathrm{rk}_1(J) + \cdots + \mathrm{rk}_r(J)$$

for all $J \subseteq [n]$. Thus by matroid partitioning, there exist sets $J_1, \ldots, J_r$ partitioning J with each J_i independent in rk_i. In fact each $J_i = \{j_i\}$ has cardinality one, since the sum of the coordinates of $\pi(v)$ is r. Then $v = \sum_i (e_{j_i}, a_{i,j_i})$.

The last remaining task is to argue that Q_{cube} induces the same regular subdivision as does $Q(A)$. Suppose v is a lower vertex of Q_{cube}, projecting to e_J for some set J. Such a v exists if J is independent in the transversal matroid of G_A. We have just shown that v may be written as a sum of vertices of the Q_i:

$$v = \sum_{i=1}^{r} (e_{\sigma(i)}, a_{i,\sigma(i)})$$

for some set function $\sigma : [r] \to [n]$. Since the $e_{\sigma(i)}$ sum to e_J, the $\sigma(i)$ must be the elements of J in some order, i.e. σ is a bijection $[r] \to J$. If v is to be a lower vertex, its last coordinate $\sum_{i=1}^{r} a_{i,\sigma(i)}$ must be minimal. These two conditions were the definition of p_J, so in fact we have $v = (e_J, p_J)$. Thus Q_{cube} and $Q(A)$ have the same lower vertices. Since every lower face is a convex hull of lower vertices, they have the same lower faces, and therefore induce the same regular subdivision, which is a matroid subdivision. □

This permits several generalisations. The first is to other matroid unions.

Proposition 6.7. *Let $M_1, \ldots, M_k$ be matroids on ground set $[n]$, and $Q_1, \ldots, Q_k$ be polytopes inducing regular matroid subdivisions of their base polytopes. If the matroid union M of $M_1, \ldots, M_k$ has rank $\mathrm{rk}(M_1) + \cdots + \mathrm{rk}(M_k)$, then*

$$(Q_1 + \cdots + Q_k) \cap \pi^{-1}([0,1]^n)$$

induces a regular matroid subdivision of the base polytope of M.

The subdivision of M may be nontrivial even if the subdivisions of $M_1, \dots, M_k$ are trivial. In Proposition 6.1, the case where $M_1, \dots, M_k$ have rank 1, we are always in the setting where their subdivisions are trivial: there are no nontrivial subdivisions of a simplex which introduce no new vertices.

Arbitrary matroid unions need not be realisable, nor even must unions meeting the conditions in Corollary 2.5, so this construction allows us to subdivide more matroid polytopes than does Proposition 6.1.

6.2. **A potentially non-regular variation.** The subdivision $S(Q)$ is of a kind known in the literature as a mixed subdivision. A generalisation to other mixed subdivisions is natural in this light, proceeding via the so-called *Cayley trick*. See Section 9.2 of [3] as a reference for this section.

We first describe the *Cayley embedding*. Given any collection $P_1, \dots, P_k$ of polytopes in a real vector space V, define a polytope in $V \times \mathbb{R}^k$,

$$\mathrm{Cayley}(P_1, \dots, P_k) = \mathrm{conv}\{P_i \times \{e_i\} : i = 1, \dots, k\}.$$

The key feature of this construction is that the intersection of $\mathrm{Cayley}(P_1, \dots, P_k)$ with the linear space $V \times (\frac{1}{k}, \dots, \frac{1}{k})$ is the Minkowski sum $P_1 + \cdots + P_k$ dilated by $1/k$.

Let S be a subdivision of $P = \mathrm{Cayley}(P_1, \dots, P_k)$. It is not difficult to see that intersecting each cell of S with a polyhedral set T gives another subdivision of $P \cap T$. In particular, intersecting S with the space $V \times (\frac{1}{k}, \dots, \frac{1}{k})$, and then dilating by k, gives a subdivision of the Minkowski sum $P_1 + \cdots + P_k$.

The subdivisions that arise in this way can be precisely characterised.

Proposition 6.8 (The Cayley trick)**.** *The intersection construction just described puts subdivisions of* $\mathrm{Cayley}(P_1, \dots, P_k)$ *in bijections with pairs* (S, f) *where* S *is a subdivision of* $P_1 + \cdots + P_k$*, and* $m : S \to \{k\text{-tuples of polytopes}\}$ *is a function so that, for all* $F \in S$*,*

(1) *every vertex of* $m(F)_i$ *is a vertex of* P_i*;*
(2) $F = m(F)_1 + \cdots + m(F)_k$*;*
(3) *if* G *is the face of* F *maximising a functional* f*, then* $m(G)_i$ *is the face of* $m(F)_i$ *maximising* f*.*

The data (S, m) *is called a* mixed subdivision*.*

We omit the proof.

In fact, the Cayley trick carries regular subdivisions of $\mathrm{Cayley}(P_1, \dots, P_k)$ to regular subdivisions of the Minkowski sum. For if $\widetilde{Q}$ is a polytope in $V \times \mathbb{R}^k \times \mathbb{R}$, then the lower faces of $\widetilde{Q} \cap (V \times (\frac{1}{k}, \dots, \frac{1}{k}) \times \mathbb{R})$ are intersections of lower faces of $\widetilde{Q}$ with $(\frac{1}{k}, \dots, \frac{1}{k})$, and projection eliminating the last coordinate commutes with this intersection.

This is precisely how the subdivision of Proposition 6.1 arises, with $k = r$ and

$$\widetilde{Q} = \mathrm{conv}\{(e_j, e_i, a_{ij}) : a_{ij} \neq \infty\}.$$

To see this, let Q be the r-th dilate of $\widetilde{Q} \cap (\mathbb{R}^E \times (\frac{1}{r}, \dots, \frac{1}{r}k) \times \mathbb{R})$, regarded as a polytope simply in $\mathbb{R}^n \times \mathbb{R}$ with the uninformative middle r coordinates discarded. By Proposition 6.8 we get that each vertex of $S(Q)$ is a Minkowski sum of vertices of the P_i, which in our case are the rank 1 matroid base polytopes $\mathrm{conv}\{e_j : a_{ij} \neq \infty\}$.

So to check Q agrees with the one constructed in the proof of Proposition 6.1, it is enough to check the heights of the vertex lying over e_J for each $J \subseteq [n]$ of size r. Let us write this vertex, as a scaled convex combination $\sum c_{ij}(e_j, e_i, a_{ij})$ of the vertices of $\widetilde{Q}$, scaled so that $\sum_{i,j} c_{ij} = r$. Clearly $c_{ij} = 0$ for $j \notin J$, since these coordinates are nonnegative on $\widetilde{Q}$. Then the conditions on the remaining c_{ij} are that $\sum_{i=1}^{r} c_{ij} = 1$ for each $j \in J$ and $\sum_{j \in J} c_{ij} = 1$ for each i: i.e. the c_{ij} define a point in the *Birkhoff polytope*. But the vertices of the Birkhoff polytope are the permutation matrices. Our task being to minimise the functional defined by the a_{ij} over the Birkhoff polytope, this minimum occurs at a vertex, that is where $c_{ij} = 1$ if $j = \sigma(i)$ and $c_{ij} = 0$ otherwise for a bijection $\sigma : [r] \to J$. This agrees with the definition of p_J.

It is not unreasonable to expected that carrying out this procedure beginning with a nonregular subdivision of $\pi(\widetilde{Q})$ will yield a nonregular subdivision. I have not checked that this is in fact the case. The Cayley trick carries nonregular subdivisions to nonregular subdivisions, but this does not account for the intersection with the unit cube. At any rate, there do exist nonregular subdivisions of products of two simplices, which are the polytopes $\operatorname{conv}\{(e_i, e_j) : i \in [r], j \in [n]\}$ from which the construction begins.

Proposition 6.9 ([3, Thm 6.2.18])**.** *The product of a simplex with r vertices and a simplex with n vertices has nonregular triangulations if and only if $1/n + 1/r \leq 1/2$, i.e. if and only if (n, r) is coordinatewise at least $(3, 6)$ or $(4, 4)$ or $(6, 3)$.*

7. Valuated matroids

Our next generalisation is linear-algebraic in nature. We begin by souping up our realisability argument for transversal matroids, using matrices of indeterminates, so that it will encode data about a linear functional.

7.1. **Valued fields.** Let $\mathbb{K}$ be a field. A *nonarchimedean valuation* on $\mathbb{K}$ is a function $\operatorname{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ satisfying three conditions:

(1) $\operatorname{val}(0) = \infty$, while $\operatorname{val}(x) \neq \infty$ for x nonzero;
(2) $\operatorname{val}(xy) = \operatorname{val}(x) + \operatorname{val}(y)$;
(3) $\operatorname{val}(x + y) \geq \min\{\operatorname{val}(x), \operatorname{val}(y)\}$.

Since we will have no use for archimedean valuations, we omit the word "nonarchimedean" in what follows.

It is immediate from the definition that $\operatorname{val}(1) = \operatorname{val}(-1) = 0$ and hence $\operatorname{val}(-x) = \operatorname{val}(x)$. It follows that if $x_1, \ldots, x_k$ are any number $k \geq 2$ of elements of $\mathbb{K}$ summing to zero, then there is a tie for the minimum value of $\operatorname{val}(x_i)$, for if instead $\operatorname{val}(x_i) < \operatorname{val}(x_j)$ for all j other than i, this contradicts

$$\operatorname{val}(x_i) = \operatorname{val}(-x_i) \geq \min\{\operatorname{val}(x_1), \ldots, \widehat{\operatorname{val}(x_i)}, \ldots, \operatorname{val}(x_n)\}$$

from repeated applications of property (3) in the definition.

Example 7.1. Let $\mathbb{K} = \mathbb{Q}$ and p be any prime. The p-adic valuation is given on nonzero rationals by

$$\operatorname{val}\left(\frac{p^e m}{n}\right) = e,$$

where m and n are integers relatively prime to p.

Example 7.2. Let $\mathbf{k}$ be a field and $\mathbb{K} = \mathbf{k}[\![t]\!][t^{-1}]$ the field of formal Laurent series in an indeterminate t with coefficients in $\mathbf{k}$: these are the power series of form

$$f = \sum_{i=\nu}^{\infty} c_i t^i$$

where ν is some integer. If we choose ν so that $c_\nu \neq 0$, then setting $\operatorname{val}(f) = \nu$ defines a valuation on $\mathbb{K}$.

We extend the last example to cast the proof of Proposition 6.5 in this light. Let $A = (a_{ij})$ and p_J be as in the previous section. Let $\mathbb{K}$ be the algebraic extension $\mathbf{k}(x_{ij})$, and define a new matrix $\widetilde{A}$ whose (i,j)th entry is $x_{ij}t^{a_{ij}}$. Then the valuation of the minor $\det \widetilde{A}_J$ is the p_J defined by the minimal assignment problem earlier.

Observe that, in the matrix A from the proof of Proposition 6.5, the valuation of the minor $\det(A_J)$ of the submatrix A_J on column set J is equal to p_J. Our next theorem expresses the fact that this construction did not depend on the very special structure of A.

Theorem 7.3. *Let $\mathbb{K}$ be any valued field and A an $r \times n$ matrix over $\mathbb{K}$ of rank r. Let $p_J = \operatorname{val}(\det A_J)$ for each $J \subseteq [n]$ of size r. Then the regular subdivision induced by the polytope*

$$Q_A := \operatorname{conv}\{(e_J, p_J) : p_J \neq \infty\}$$

is a matroid subdivision.

Conjecture 7.4 (Sturmfels)**.** *If M is a graphic matroid, then every subdivision of M arises from the construction of Theorem 7.3.*

7.2. **Valuated matroids.** Valuated matroids were introduced by Dress and Wenzel [4] as part of a programme to unify matroids and oriented matroids; they viewed the latter as reflecting a "coefficient domain" $\{+, 0, -\}$ and generalised to other classes of coefficients. (They made other generalisations simultaneously, such as infinite ground sets.) Valuated matroids were later rediscovered in the context of tropical geometry and called *tropical Plücker vectors* by Speyer [16].

Definition 7.5. A vector (p_J) of numbers in $\mathbb{R} \cup \{\infty\}$, indexed by subsets J of $[n]$ of cardinality r, is a *valuated matroid* if, for every subset S of $[n]$ of cardinality $r-2$ and elements $a, b, c, d \in [n] \setminus S$, there is a tie for the minimum value among the three sums

$$p_{S\cup\{a,b\}} + p_{S\cup\{c,d\}}, p_{S\cup\{a,c\}} + p_{S\cup\{b,d\}}, p_{S\cup\{a,d\}} + p_{S\cup\{b,c\}}. \tag{7.1}$$

Punctuation in subscripts is cumbrous, so we will allow ourselves to write p_{Kab} instead of $p_{K\cup\{a,b\}}$ and so on.

Proposition 7.6 (Kapranov [10]; also [16, Prop 2.2])**.** *The regular subdivision induced by*

$$Q_A := \operatorname{conv}\{(e_J, p_J) : p_J \neq \infty\}$$

is a matroid subdivision if and only if (p_J) is a valuated matroid.

Proof. Let (p_J) be a valuated matroid. We will show that being a valuated matroid is equivalent to the hypothesis of Corollary 5.2, that for any edge

$$\operatorname{conv}\{(e_J, p_J), (e_K, p_K)\}$$

of Q_A, the number of elements $\ell = |J \setminus K| = |K \setminus J|$ in the differences is 1. Suppose otherwise. The case $\ell = 2$ is ruled out by Example 5.3. We can write $J = S \cup \{a, b\}$, $K = S \cup \{c, d\}$ for some distinct $a, b, c, d \in [n] \setminus S$, and then $e_{Sab}, e_{Sac}, e_{Sad}, e_{Sbc}, e_{Sbd}, e_{Scd}$ are the vertices of an octahedral face of $\pi(Q_A)$; the valuated matroid axiom precisely ensures that the edge $\operatorname{conv}\{e_{Sab}, e_{Scd}\}$ does not occur.

If $\ell \geq 3$, we work in the projection $\pi(Q_A)$. Let the bad edge in the projection be $\mathcal{E} = \operatorname{conv}\{e_{S\cup T}, e_{S\cup U}\}$. The vertices e_K of $\pi(Q_A)$ with $S \subseteq K \subseteq S \cup T \cup U$ make up a face F of $\pi(Q_A)$ of dimension at least 2, and within F the edge $\mathcal{E}$ is contained in a face G of dimension 2. Of the remaining edges of G, there are no others $\operatorname{conv}\{e_J, e_K\}$ with $|J \setminus K| = |K \setminus J| = \ell$, as any such edge would intersect $\mathcal{E}$ at their common midpoint. So all the remaining edges are shorter; by induction on ℓ they are all in directions $e_i - e_j$. But these directions span a two-dimensional space, and one sees readily that the only possibilities are translates of spaces of the forms $\operatorname{span}\{e_i - e_j, e_i - e_k\}$ and $\operatorname{span}\{e_i - e_j, e_k - e_h\}$, neither of which can hold our edge $\mathcal{E}$ with $\ell \geq 3$.

Example 5.3 also establishes the inverse, that if (p_J) is not a valuated matroid then Q_A has a bad edge. Indeed, if the valuated matroid axiom fails for S, a, b, c, d, then restricting to the corresponding face we find a long edge in Q_A. □

Proof of Theorem 7.3. It is enough to show that the (p_J) are a valuated matroid, by Proposition 7.6. This they do because the maximal minors of any matrix A satisfy the *quadratic Plücker relations*:

$$\det A_{Sab} \det A_{Scd} - \det A_{Sac} \det A_{Sbd} + \det A_{Sad} \det A_{Sbc} = 0.$$

These relations hold for a 2×4 matrix (check this yourself!) and larger cases can be reduced to this one by row operations which partially diagonalise along the columns in J. These relations imply the needed condition on the valuations $p_J = \operatorname{val}(\det A_J)$ by the discussion after the definition. □

We pause here to show that every matroid can appear as a top-dimensional cell of a matroid subdivision.

Example 7.7. Let rk be a matroid rank function, with $\operatorname{rk}(E) = r$. Then setting

$$p_J = -\operatorname{rk}(J)$$

for each $J \subseteq E$ of size r gives a valuated matroid.

Here is one argument for this: let E' be a disjoint copy of E, and let rk' be the matroid attained from rk by adjoining all the elements of E' freely (without increasing the rank). Project the base polytope of rk' to a polytope $Q \subseteq \mathbb{R}^E \times \mathbb{R}$ by mapping an unprimed e_i to $(e_i, -1)$ and a primed e'_i to $(e_i, 0)$. The edges of Q are projections of the edges of $P_{\text{base}}(\operatorname{rk}')$, so they are still in the directions mandated by Corollary 5.2 and Q gives a regular subdivision of polymatroids. By the argument of the proof of Proposition 6.1, intersecting Q with $[0, 1]^E \times \mathbb{R}$ gives a polytope inducing a regular matroid subdivision, and its vertices are projections of vertices of Q. The lowest of these that projects to p_J is obtained by taking the largest subset of J possible from E, namely $\operatorname{rk}(J)$ elements; and the remainder from E'; the height of this vertex is thus $-\operatorname{rk}(J)$.

Its associated regular subdivision has the base polytope of rk as a face, namely the face determined by all vertices at height $p_J = -r$ (which is the minimum height).

7.3. **Lafforgue's theorem.** A tiny fragment of the work for which Lafforgue was recognised with the Fields medal gives a condition for a matroid to possess a subdivision.

Suppose a matroid M on ground set E has a linear realisation A over some field $\mathbf{k}$. Regard A as a vector configuration $(a_i : i \in E)$, where each a_i is an element of $\mathbf{k}^r$. There are systematic ways to alter A without affecting the matroid it realises: two such are to apply a linear transformation to the whole vector configuration, or to multiply individual vectors by nonzero scalars. To say that again, there is an action of $\mathrm{GL}_r(\mathbf{k}) \times (\mathbf{k}^*)^E$ on the space of matrices $\mathbb{K}^{r\times E}$ whence A is drawn, the first factor acting on the left and the second on the right; orbits under this action realise the same matroid. We regard these deformations of the realisation as trivial.

Theorem 7.8 (Lafforgue)**.** *Any matroid with a linear realisation that can be nontrivially continuously deformed while remaining a realisation has a nontrivial matroid subdivision.*

Lafforgue's original version of the statement spoke not of continuity but of the dimension of the algebraic variety of realisations of M. For the above statement, with its talk of continuity, we had best take the base field of the realisation to be $\mathbb{C}$.

Quasi-proof. Express the coordinates of the vectors in the continuous deformation as power series in a variable t, giving a matrix $A(t)$ of power series. If this realisation is deformed trivially, the effect on the maximal minors is to replace $\det A_J$ with $\det A_J \cdot \prod_{j\in J} u_j$ for some vector u of length $|E|$, corresponding to the rescalings of individual vectors. In particular, if $\sum c_J e_J = 0$ is any affine-linear relation among the vertices of the base polytope, the quantity $\pi := \prod(\det A_J)^{c_J}$ is invariant under trivial deformations. The converse also holds: since our deformation is nontrivial, we know some such product of powers of minors of $A(t)$ is not unity.

Let us suppose that we can find some t_0 such that π has a zero or pole at t_0. (This is what makes the proof quasi: it's fine e.g. for rational functions, but in general there are questions of convergence and of encountering functions like exp.) Then there does not exist a vector v and constant c such that the order of vanishing of A_J at t_0 equals $c + \sum_{j\in J} v_j$ for all J. Then, if val is the valuation of Example 7.2, the function $f(t) \mapsto \mathrm{val}(f(t-t_0))$ is also a valuation, so it induces a matroid subdivision by Theorem 7.3. The condition that v not exist ensures precisely that the vertices of the polytope Q constructed do not lie in a single hyperplane, so the subdivision is nontrivial. □

Example 7.9. We choose the *Desargues matroid* as the substrate of an example. The Desargues matroid is quickest defined as the truncation of the graphic matroid of K_5 to rank 3, and we will use this to name its ground set elements, $\{12, 13, \ldots, 45\}$. But it's more classically viewed as the matroid of the configuration of points and lines in Desargues' theorem of projective geometry:

> Two triangles in the plane are in perspective from a point if and only if they are in perspective from a line.

This theorem holds over any field (and indeed even any skewfield). As such, the Desargues matroid has a wealth of realisations over any large enough field, whereas the *non-Desargues matroid* obtained by declaring one of its triples of dependent elements independent has none, as a realisation would be a configuration contradicting Desargues' theorem.

Let us parametrise the space of realisations of the Desargues matroid, modulo trivial motions. The automorphism group of $\mathbb{P}^2$ acts transitively on quadruples of noncollinear points, so we may assume that four of the vectors are fixed: let us place the perspector 12 at $(1,1,1)$ and the first triangle 13, 14, 15 at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The vertices of the second triangle vary along the lines defined through the perspector; we will give 23, 24, 25 the coordinates $(1,t,t)$, $(u,1,u)$, $(v,v,1)$, where t,u,v are not 0 or 1 (but may be ∞). Intersecting the pairs of sides of the remaining triangles gives the coordinates for the remaining points. Altogether the realisation is

$$\begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & u & v & tu-u & -tv+v & 0 \\ 1 & 0 & 1 & 0 & t & 1 & v & -tu+t & 0 & uv-v \\ 1 & 0 & 0 & 1 & t & u & 1 & 0 & tv-t & -uv+u \end{pmatrix}.$$

To apply the construction of our proof we need to pick a continuous motion with a single parameter in this space. Let us take the one where $(t,u,v) = (t,t^2,t^4)$. This substitution makes the realisation

$$\begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & t^2 & t^4 & t^3-t^2 & -t^5+t^4 & 0 \\ 1 & 0 & 1 & 0 & t & 1 & t^4 & -t^3+t & 0 & t^6-t^4 \\ 1 & 0 & 0 & 1 & t & t^2 & 1 & 0 & t^5-t & -t^6+t^2 \end{pmatrix}.$$

Since the matroid represented by these vectors changes when t becomes 0 (even though we do not let the columns 34, 35, 45 become entirely zero but rather divide through by the smallest power of t appearing first), we might expect that the valuations in t of the 3×3 minors of this matrix will already give a nontrivial subdivision, and in fact this is correct.

From here it is just a matter of using your favourite computer algebra package (mine's Macaulay2) to write down the valuations of all determinants of this matrix, and your favourite discrete geometry package (mine's polymake) to compute the regular subdivision. It has fourteen maximal-dimensional faces. Here are the lists of bases of each:

12,13,14 12,13,15 12,14,15 13,14,15 12,14,23 12,15,23 14,15,23 12,13,24 12,15,24 13,15,24 12,23,24 15,23,24 12,13,25 12,14,25 13,14,25 12,23,25 14,23,25 12,24,25 13,24,25 23,24,25 12,13,34 12,15,34 13,15,34 12,23,34 15,23,34 12,25,34 13,25,34 23,25,34 12,13,35 12,14,35 13,14,35 12,23,35 14,23,35 12,24,35 13,24,35 23,24,35 12,34,35 13,34,35 23,34,35 12,13,45 12,14,45 13,14,45 12,23,45 14,23,45 12,24,45 13,24,45 23,24,45 12,34,45 13,34,45 23,34,45

12,13,14 12,13,15 13,14,15 12,14,23 13,14,23 12,15,23 13,15,23 14,15,23 12,13,24 13,15,24 12,23,24 13,23,24 15,23,24 12,13,25 13,14,25 12,23,25 13,23,25 14,23,25 13,24,25 23,24,25 12,13,34 13,15,34 12,23,34 13,23,34 15,23,34 13,25,34 23,25,34 12,13,35 13,14,35 12,23,35 13,23,35 14,23,35 13,24,35 23,24,35 13,34,35 23,34,35 12,13,45 13,14,45 12,23,45 13,23,45 14,23,45 13,24,45 23,24,45 13,34,45 23,34,45

12,13,14 13,14,15 12,14,23 13,14,23 14,15,23 12,13,24 13,14,24 13,15,24 12,23,24 13,23,24 14,23,24 15,23,24 13,14,25 14,23,25 13,24,25 23,24,25 12,13,34 12,14,34 13,15,34 14,15,34 12,23,34 13,23,34 14,23,34 15,23,34 12,24,34 13,24,34 14,24,34 15,24,34 13,25,34 14,25,34 23,25,34 24,25,34 13,14,35 14,23,35 13,24,35 23,24,35 13,34,35 14,34,35 23,34,35 24,34,35 13,14,45 14,23,45 13,24,45 23,24,45 13,34,45 14,34,45 23,34,45 24,34,45

12,13,14 12,14,15 13,14,15 12,14,23 14,15,23 12,13,24 13,14,24 12,15,24 13,15,24 14,15,24 12,23,24 14,23,24 15,23,24 12,14,25 13,14,25 14,23,25 12,24,25 13,24,25 14,24,25 23,24,25 12,14,34 14,15,34 12,24,34 14,24,34

15,24,34 14,25,34 24,25,34 12,14,35 13,14,35 14,23,35 12,24,35 13,24,35 14,24,35 23,24,35 14,34,35 24,34,35 12,14,45 13,14,45 14,23,45 12,24,45 13,24,45 14,24,45 23,24,45 14,34,45 24,34,45

12,13,14 12,14,15 13,14,15 12,14,23 14,15,23 12,13,24 12,15,24 13,15,24 12,23,24 15,23,24 12,14,25 13,14,25 14,23,25 12,24,25 13,24,25 23,24,25 12,13,34 12,14,34 12,15,34 13,15,34 14,15,34 12,23,34 15,23,34 12,24,34 15,24,34 12,25,34 13,25,34 14,25,34 23,25,34 24,25,34 12,14,35 13,14,35 14,23,35 12,24,35 13,24,35 23,24,35 12,34,35 13,34,35 14,34,35 23,34,35 24,34,35 12,14,45 13,14,45 14,23,45 12,24,45 13,24,45 23,24,45 12,34,45 13,34,45 14,34,45 23,34,45 24,34,45

12,13,15 13,14,15 12,15,23 13,15,23 14,15,23 13,15,24 15,23,24 12,13,25 13,14,25 12,23,25 13,23,25 14,23,25 13,24,25 23,24,25 13,15,34 15,23,34 13,25,34 23,25,34 12,13,35 13,14,35 12,23,35 13,23,35 14,23,35 13,24,35 23,24,35 13,34,35 23,34,35 12,13,45 13,14,45 13,15,45 12,23,45 13,23,45 14,23,45 15,23,45 13,24,45 23,24,45 13,25,45 23,25,45 13,34,45 23,34,45 13,35,45 23,35,45

12,13,15 12,14,15 13,14,15 12,15,23 14,15,23 12,15,24 13,15,24 15,23,24 12,13,25 12,14,25 13,14,25 12,23,25 14,23,25 12,24,25 13,24,25 23,24,25 12,15,34 13,15,34 15,23,34 12,25,34 13,25,34 23,25,34 12,13,35 12,14,35 13,14,35 12,15,35 14,15,35 12,23,35 14,23,35 12,24,35 13,24,35 15,24,35 23,24,35 12,25,35 14,25,35 24,25,35 12,34,35 13,34,35 15,34,35 23,34,35 25,34,35 12,15,45 13,15,45 15,23,45 12,25,45 13,25,45 23,25,45 12,35,45 13,35,45 15,35,45 23,35,45 25,35,45

12,14,15 13,14,15 14,15,23 12,15,24 13,15,24 14,15,24 15,23,24 12,14,25 13,14,25 14,23,25 12,24,25 13,24,25 14,24,25 23,24,25 14,15,34 15,24,34 14,25,34 24,25,34 12,14,35 13,14,35 14,15,35 14,23,35 12,24,35 13,24,35 14,24,35 15,24,35 23,24,35 14,25,35 24,25,35 14,34,35 24,34,35 12,14,45 13,14,45 14,23,45 12,24,45 13,24,45 14,24,45 23,24,45 14,34,45 24,34,45 14,35,45 24,35,45

12,13,15 13,14,15 12,15,23 13,15,23 14,15,23 13,15,24 15,23,24 12,13,25 13,14,25 13,15,25 12,23,25 13,23,25 14,23,25 15,23,25 13,24,25 23,24,25 13,15,34 15,23,34 13,25,34 23,25,34 12,13,35 13,14,35 12,15,35 14,15,35 12,23,35 13,23,35 14,23,35 15,23,35 13,24,35 15,24,35 23,24,35 12,25,35 13,25,35 14,25,35 15,25,35 24,25,35 13,34,35 15,34,35 23,34,35 25,34,35 13,15,45 15,23,45 13,25,45 23,25,45 13,35,45 15,35,45 23,35,45 25,35,45

12,13,15 12,14,15 13,14,15 12,15,23 14,15,23 12,15,24 13,15,24 15,23,24 12,13,25 12,14,25 13,14,25 12,23,25 14,23,25 12,24,25 13,24,25 23,24,25 12,15,34 13,15,34 15,23,34 12,25,34 13,25,34 23,25,34 12,13,35 12,14,35 13,14,35 12,23,35 14,23,35 12,24,35 13,24,35 23,24,35 12,34,35 13,34,35 23,34,35 12,13,45 12,14,45 13,14,45 12,15,45 13,15,45 12,23,45 14,23,45 15,23,45 12,24,45 13,24,45 23,24,45 12,25,45 13,25,45 23,25,45 12,34,45 13,34,45 23,34,45 12,35,45 13,35,45 23,35,45

12,13,15 12,14,15 13,14,15 12,15,23 14,15,23 12,15,24 13,15,24 15,23,24 12,13,25 12,14,25 13,14,25 13,15,25 14,15,25 12,23,25 14,23,25 15,23,25 12,24,25 13,24,25 15,24,25 23,24,25 12,15,34 13,15,34 15,23,34 12,25,34 13,25,34 15,25,34 23,25,34 12,15,35 14,15,35 15,24,35 12,25,35 14,25,35 15,25,35 24,25,35 15,34,35 25,34,35 12,15,45 13,15,45 15,23,45 12,25,45 13,25,45 15,25,45 23,25,45 15,35,45 25,35,45

12,14,15 13,14,15 14,15,23 12,15,24 13,15,24 14,15,24 15,23,24 12,14,25 13,14,25 14,15,25 14,23,25 12,24,25 13,24,25 14,24,25 15,24,25 23,24,25 14,15,34 15,24,34 14,25,34 24,25,34 14,15,35 15,24,35 14,25,35 24,25,35 12,14,45 13,14,45 12,15,45 13,15,45 14,23,45 15,23,45 12,24,45 13,24,45 14,24,45 15,24,45 23,24,45 12,25,45 13,25,45 14,25,45 15,25,45 23,25,45 14,34,45 15,34,45 24,34,45 25,34,45 14,35,45 15,35,45 24,35,45 25,35,45

12,14,15 13,14,15 14,15,23 12,15,24 13,15,24 15,23,24 12,14,25 13,14,25 14,23,25 12,24,25 13,24,25 23,24,25 12,15,34 13,15,34 14,15,34 15,23,34 15,24,34 12,25,34 13,25,34 14,25,34 23,25,34 24,25,34 12,14,35 13,14,35 14,15,35 14,23,35 12,24,35 13,24,35 15,24,35 23,24,35 14,25,35 24,25,35 12,34,35 13,34,35 14,34,35 15,34,35 23,34,35 24,34,35 25,34,35 12,14,45 13,14,45 12,15,45 13,15,45 14,23,45 15,23,45 12,24,45 13,24,45 23,24,45 12,25,45 13,25,45 23,25,45 12,34,45 13,34,45 14,34,45 15,34,45 23,34,45 24,34,45 25,34,45 12,35,45 13,35,45 14,35,45 15,35,45 23,35,45 24,35,45 25,35,45

12,14,15 13,14,15 14,15,23 12,15,24 13,15,24 15,23,24 12,14,25 13,14,25 14,15,25 14,23,25 12,24,25 13,24,25 15,24,25 23,24,25 12,15,34 13,15,34 14,15,34 15,23,34 15,24,34 12,25,34 13,25,34 14,25,34 15,25,34 23,25,34 24,25,34 14,15,35 15,24,35 14,25,35 24,25,35 15,34,35 25,34,35 12,15,45 13,15,45 15,23,45 12,25,45 13,25,45 15,25,45 23,25,45 15,34,45 25,34,45 15,35,45 25,35,45

References

[1] A. Borovik, I. Gelfand, N. White. *Coxeter Matroids*. Birkhäuser, Boston, 2003.

[2] V. Chatelain, J.L. Ramírez Alfonsin, *Matroid base polytope decomposition*, `http://arxiv.org/abs/0909.0840`.

[3] J. De Loera, J. Rambau, F. Santos, *Triangulations: Structures for Algorithms and Applications*, Algorithms and Computation in Mathematics, Springer, 2010.

[4] A. Dress, W. Wenzel, *Valuated matroids*, Adv. Math. **93** no. 2 (1992), 214–250.

[5] Jack Edmonds, *Submodular functions, matroids, and certain polyhedra*, Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969) Gordon and Breach, New York, 1970, 69–87.

[6] Eva Maria Feichtner and Bernd Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005) no. 4, 437–468.

[7] Alex Fink and Felipe Rincón, *Stiefel tropical linear spaces*, Journal of Combinatorial Theory, Series A **135** (2015), 291–331.

[8] Israel Gelfand, Mark Goresky, Robert MacPherson and Vera Serganova, *Combinatorial geometries, convex polyhedra and Schubert cells*, Adv. in Math. **63** (1987), 301–316.

[9] P. Hacking, S. Keel, and J. Tevelev. *Compactification of the moduli space of hyperplane arrangements*. J. Algebraic Geom. **15** (2006), 657–680.

[10] Mikhail M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gel'fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, 29–110.

[11] Laurent Lafforgue, *Pavages des simplexes, schémas de graphes recollés et compactification des* $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$. Invent. Math. **136** (1999), 233–271.

[12] Laurent Lafforgue, *Chirurgie des grassmanniennes*, CRM Monograph Series, 19. American Mathematical Society, Providence, RI, 2003.

[13] James G. Oxley, *Matroid theory*, 2006, Oxford University Press.

[14] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, International Mathematics Research Notices **2009**, no. 6 (2009), 1026–1106.

[15] Alexander Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2003.

[16] David Speyer, *Tropical linear spaces*, SIAM Journal on Discrete Mathematics **22** no. 4 (2008), 1527–1558. arXiv:math/0410455.

[17] Dominic J. A. Welsh, *Matroid theory*, L. M. S. Monographs, No. 8, Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976, xi+433.