Tropical cycles and Chow polytopes

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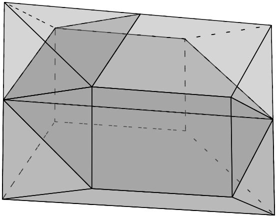
Motivation

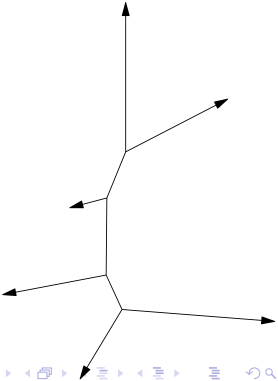
Newton polytopes give us a nice combinatorial understanding of tropical hypersurfaces, *matroid polytopes* of tropical linear spaces.

Chow polytopes are the common generalisation.

Do Chow polytopes yield a nice combinatorial understanding of tropical varieties?

$$V((t^6-t^5-t^4-t^3+t^2+t)x+(-t^6+2t^3-1)y+(-t^2-t+1)z+(t^5+t^4-t^3)w,\ (t^5-t^3-t^2+1)yz+tz^2+(t^6-t^5-t^3+t^2)yw+(-t^4+t^3-t-1)zw+(-t^6+t^4+t^3)w^2)\subseteq\mathbb{P}^3$$



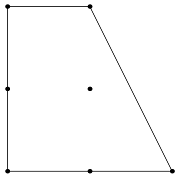


Review: Newton polytopes

Given a constant-coefficient hypersurface $V(f) \subseteq \mathbb{P}^{n-1}$, with $f \in \mathbb{K}[x_1, \ldots, x_n]$ homogeneous, $\mathrm{Trop}(X) \subseteq \mathbb{R}^n/\mathbb{1}$ is the codimension 1 part of the normal fan to the Newton polytope of f,

$$\mathrm{Newt}(f) = \mathrm{conv}\{m \in (\mathbb{Z}^n)^\vee : x^m \text{ is a monomial of } f\} \subseteq (\mathbb{R}^n)^\vee.$$

If $\mathbb{K}$ is a valued field, the valuations of the coefficients of f induce a *regular subdivision* of $\mathrm{Newt}(f)$. Use the *normal complex* to this subdivision instead.



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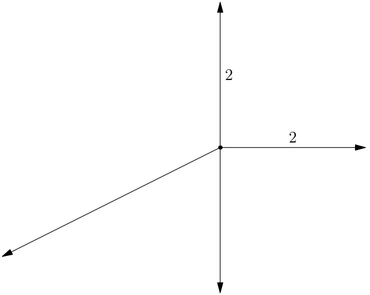

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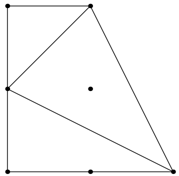

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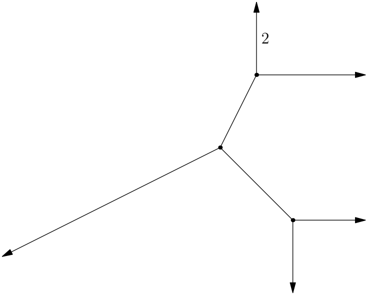


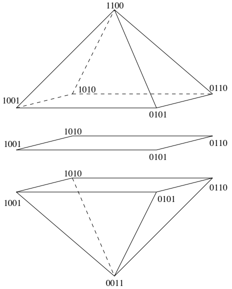
Review: Matroid polytopes

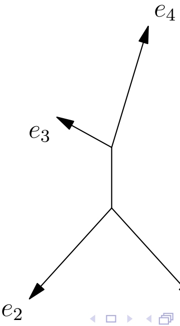
Given a constant-coefficient linear space $V(I) \subseteq \mathbb{P}^{n-1}$, with $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ a linear ideal, $\mathrm{Trop}(X) \subseteq \mathbb{R}^n/\mathbb{1}$ is the union of normals to *loop-free* faces of the matroid polytope of I,

$$Q(M_I) = \mathrm{conv}\{\textstyle\sum_{j \in J} e_J : p_J(I) \neq 0\} \subseteq (\mathbb{R}^n)^\vee,$$

where $p_J(I)$ are the *Plücker coordinates* of I.

If $\mathbb{K}$ is a valued field, the valuations of the coefficients of I induce a *regular subdivision* of $Q(M_I)$. Use the *normal complex* to this subdivision instead.





The Chow variety

What about parametrising *classical* subvarieties $X \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$? Cycles?

Definition

The Chow variety $G(d, n, r)$ is the parameter space for (effective) cycles in $\mathbb{P}^{n-1}$ of dimension $d - 1$ and degree r.

The Chow variety is projective, and has a projective embedding via the *Chow form* R_X:

$$G(d, n, r) \hookrightarrow \mathbb{P}(\mathbb{K}[G(n-d, n)]_r)$$
$$X \mapsto R_X.$$

The coordinate ring $\mathbb{K}[G(n-d, n)]$ of the Grassmannian $G(n-d, n)$ has a presentation in Plücker coordinates:

$$\mathbb{K}[G(n-d, n)] = \mathbb{K}\big[\, [J] : J \in \tbinom{[n]}{n-d} \,\big] / (\text{Plücker relations}).$$

Chow polytopes

The torus $(\mathbb{K}^*)^n$ acts on $G(d, n, r) \subseteq \mathbb{K}[G(n-d, n)]$ diagonally.

The *weight* of the bracket $[J]$ is $e_J := \sum_{j \in J} e_j$. That is,

$$(h_1, \dots, h_n) \cdot [J] = \prod_{j \in J} h_j [J].$$

The weight of a monomial $\prod_i [J_i]^{m_i}$ is $\sum_i m_i e_{J_i}$.

Definition

The Chow polytope of X, $\mathrm{Chow}(X) \subseteq (\mathbb{R}^n)^\vee$, is the *weight polytope* of its Chow form R_X:

$$\mathrm{Chow}(X) = \mathrm{conv}\{\text{weight of } m : m \text{ a monomial of } R_X\}.$$

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Examples

- For X a *hypersurface* $V(f)$, $R_X = f$ and $\mathrm{Chow}(X)$ is the *Newton polytope*.
- For X a *linear space*, $R_X = \sum_J p_J[J]$ is the linear form in the brackets with the Plücker coordinates of X as coefficients, and $\mathrm{Chow}(X)$ is the *matroid polytope* of X.
- For X an embedded *toric variety* in $\mathbb{P}^{n-1}$, $\mathrm{Chow}(X)$ is a *secondary polytope* [Gelfand-Kapranov-Zelevinsky].

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Faces of Chow polytopes

The torus action on $G(d,n,r)$ lets us take toric limits: given a one-parameter subgroup $u : \mathbb{K}^* \to (\mathbb{K}^*)^n$, send $x \in G(d,n,r)$ to $\lim_{t\to\infty} u(t)\cdot x$.

These correspond to toric degenerations of cycles in $\mathbb{P}^{n-1}$.

Theorem (Kapranov–Sturmfels–Zelevinsky)

The face poset of $\mathrm{Chow}(X)$ is isomorphic to the poset of toric degenerations of X.

In particular, the vertices of $\mathrm{Chow}(X)$ are in bijection with toric degenerations of X that are sums of coordinate $(d-1)$-planes $L_J = V(x_j = 0 : j \in J)$.
A cycle $\sum_J m_J L_J$ corresponds to the vertex $\sum_J m_J e_J$.

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Over a valued field

Suppose $(\mathbb{K}, \nu)$ is a valued field, with residue field $\mathbf{k} \hookrightarrow \mathbb{K}$.

Over $\mathbf{k}$, the torus $(\mathbf{k}^*)^n \times \mathbf{k}^*$ acts on $G(d, n, r) \subseteq \mathbb{K}[G(n-d, n)]$: brackets $[J]$ have weight $(e_J, 0)$, and $a \in \mathbb{K}$ has weight $(0, \nu(a))$.

For a cycle $X \subseteq \mathbb{P}^{n-1}$ this gives us a weight polytope $\Pi \subseteq (\mathbb{R}^{n+1})^\vee$. Its vertices are the vertices of $\mathrm{Chow}(X)$, lifted according to ν.

Definition

The Chow subdivision $\mathrm{Chow}'(X)$ of X is the regular subdivision of $\mathrm{Chow}(X)$ induced by the lower faces of Π.

Examples: Newton and matroid polytope subdivisions.

The tropical side

Fact

$\mathrm{Trop}(X)$ *is a subcomplex of the normal complex of* $\mathrm{Chow}'(X)$.

$\mathrm{Trop}(X)$ determines $\mathrm{Chow}'(X)$, by orthant-shooting.

Let σ_J be the cone in $\mathbb{R}^n/\mathbb{1}$ with generators $\{e_j : j \in J\}$.
For a 0-dimensional tropical variety C, let $\#C$ be the sum of the multiplicities of the points of C.

Theorem (Dickenstein–Feichtner–Sturmfels, F)

Let $u \in \mathbb{R}^n$ *be s.t.* $\mathrm{face}_u \mathrm{Chow}'(X)$ *is a vertex. Then*

$$\mathrm{face}_u \mathrm{Chow}'(X) = \sum_{J \in \binom{[n]}{n-d}} \#([u + \sigma_J] \cdot \mathrm{Trop}\, X) e_J.$$

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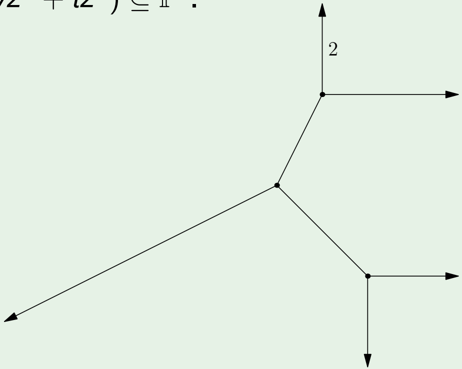
Orthant-shooting

When X is a hypersurface, this is just *ray-shooting*.

Example

Here $X = V(xy^2 + x^2z + t^2y^2z + yz^2 + tz^3) \subseteq \mathbb{P}^2$.



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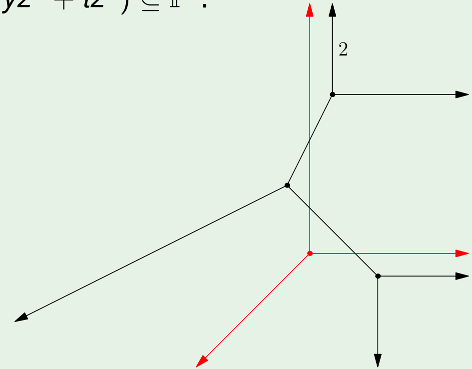

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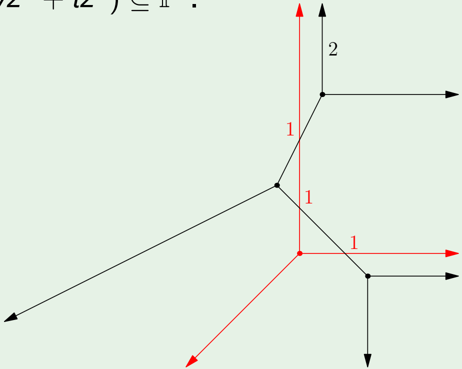

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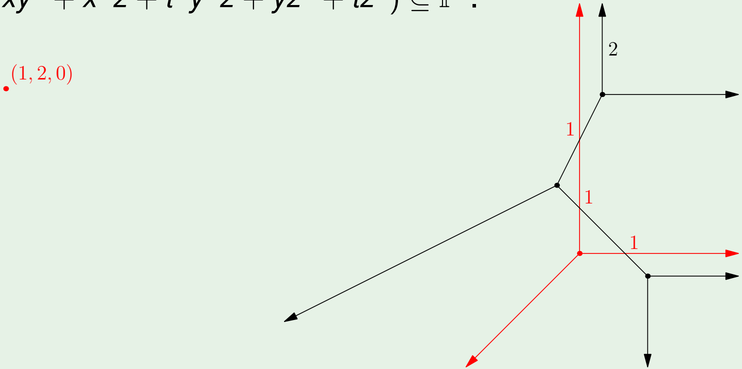

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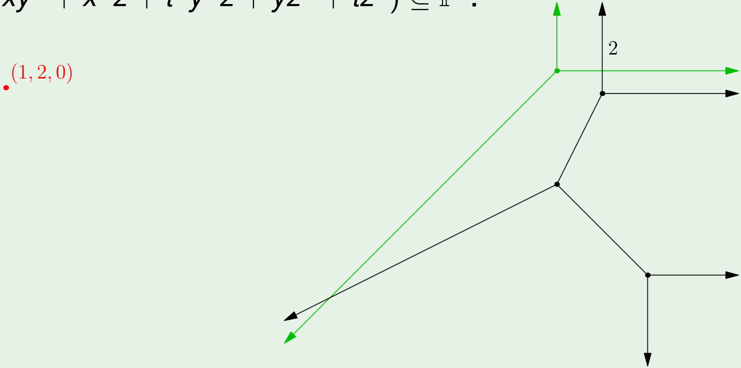

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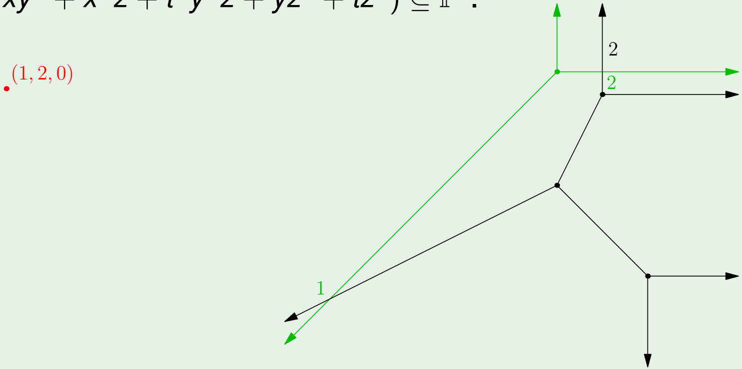

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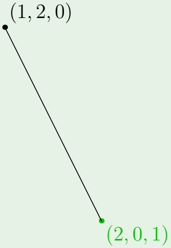

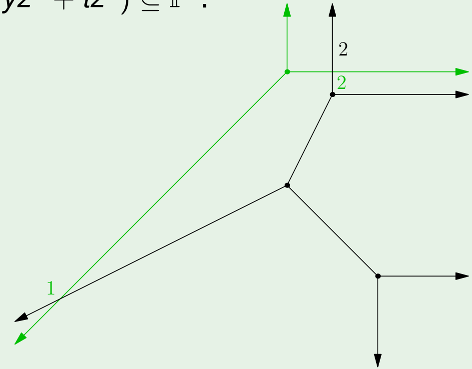

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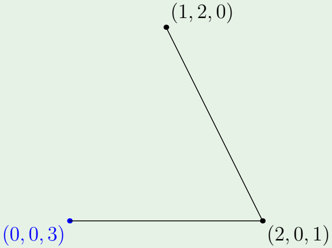

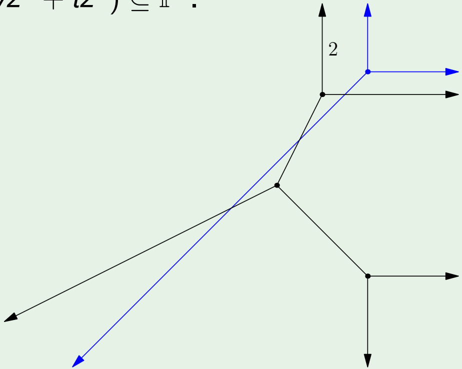

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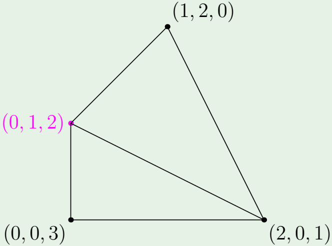

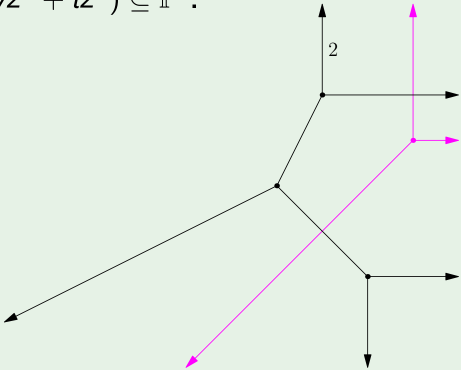

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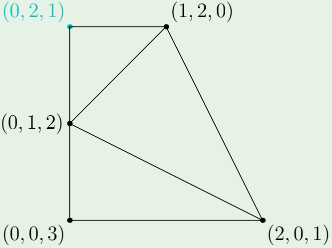
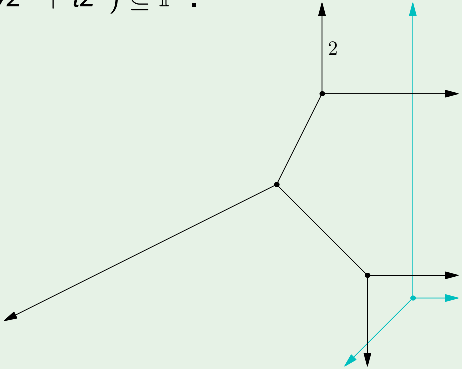
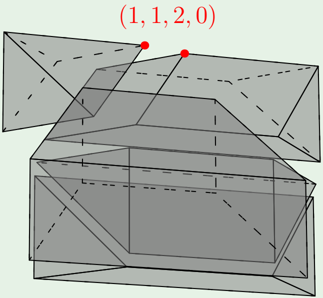


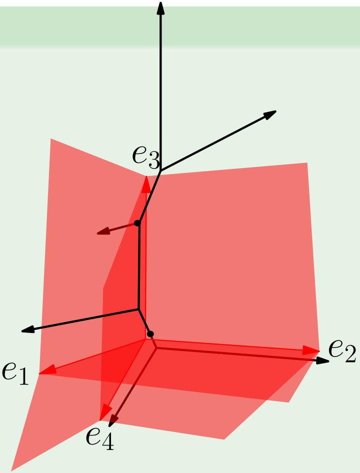


Orthant-shooting

In general we shoot higher-dimensional cones:

Examples





Tropical cycles

Until now we've only considered tropicalisations. We'd like to work with abstract tropical objects:

Definition

A tropical cycle in $\mathbb{R}^{n-1} = \mathbb{R}^n/\mathbb{1}$ is an element of

$$\left\{\begin{array}{l}\text{pure integral polyhedral complexes w/ integer}\\ \text{weights satisfying the balancing condition}\end{array}\right\} \Big/ \left(\begin{array}{l}\text{refinement of}\\ \text{complexes}\end{array}\right).$$

A tropical variety is a tropical cycle with all weights nonnegative.
It is a fan cycle if the underlying complex is a fan.

Let Z^i be the $\mathbb{Z}$-module of tropical cycles in $\mathbb{R}^{n-1}$ of *codimension* i, and $Z^* = \bigoplus_i Z^i$.

For Σ a polyhedral complex, let $Z^*(\Sigma)$ be the finite-dimensional submodule of cycles whose facets *are* faces of Σ. (This is not Kaiserslautern notation!)

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The intersection product on tropical cycles

For tropical cycles C and D, let $C \cdot D$ denote the *(stable) intersection* of tropical intersection theory: $C \cdot D = \lim_{\epsilon \to 0} C \cap (\text{D displaced by } \epsilon)$ with lattice multiplicities.
Stable intersection makes Z^* into a graded ring.

Fan tropical cycles and their intersection product make other appearances:

- as elements of the direct limit of Chow cohomology rings of toric varieties [Fulton-Sturmfels];
- as *Minkowski weights*, one representation of the elements of the *polytope algebra* of Peter McMullen.

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The stable Minkowski sum

The *Minkowski sum* of sets $S, T \subseteq \mathbb{R}^{n-1}$ is $\{s + t : s \in S, t \in T\}$.

For $\sigma \subseteq \mathbb{R}^{n-1}$, let $N_\sigma = \mathbb{Z}^{n-1} \cap$ the $\mathbb{R}$-subspace generated by a translate of σ containing 0. Define multiplicities

$$\mu_{\sigma,\tau} = \begin{cases} [N_{\sigma+\tau} : N_\sigma + N_\tau] & \text{if } \dim(\sigma+\tau) = \dim\sigma + \dim\tau \\ 0 & \text{otherwise.} \end{cases}$$

This is the same as for tropical intersection, except for the condition.

Definition

The stable Minkowski sum $C \boxplus D$ of tropical cycles C and D is their Minkowski sum with the right multiplicities: for every facet σ of C with mult m_σ and τ of D with mult m_τ, $C \boxplus D$ has a facet $\sigma + \tau$ with mult $\mu_{\sigma,\tau} m_\sigma m_\tau$.

Proposition

The stable Minkowski sum of tropical cycles is a tropical cycle.

Orthant-shooting revisited

Definition

Let the tropical Chow hypersurface of $X \subseteq \mathbb{P}^{n-1}$ be the codimension 1 part of the normal complex to $\mathrm{Chow}'(X)$.

Let $\mathcal{L}$ be the canonical tropical hyperplane $\mathrm{Trop}\, V(x_1 + \ldots + x_n)$, and $\mathcal{L}_{(i)}$ its dimension i skeleton.
For a tropical cycle X let X^{refl} be the reflection of X through the origin.

Main theorem 1 (F)

Let X be a codimension k cycle in $\mathbb{P}^{n-1}$. The tropical Chow hypersurface of X is $\mathrm{Trop}(X) \boxplus (\mathcal{L}_{(k-1)})^{\mathrm{refl}}$.

Definition

Define $ch : Z^k \to Z^1$ by $ch(C) = C \boxplus (\mathcal{L}_{(k-1)})^{\mathrm{refl}}$ for a tropical cycle C.

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Let the tropical Chow hypersurface of $X \subseteq \mathbb{P}^{n-1}$ be the codimension 1 part of the normal complex to $\mathrm{Chow}'(X)$.

Let $\mathcal{L}$ be the canonical tropical hyperplane $\mathrm{Trop}\, V(x_1 + \ldots + x_n)$, and $\mathcal{L}_{(i)}$ its dimension i skeleton.
For a tropical cycle X let X^{refl} be the reflection of X through the origin.

Main theorem 1 (F)

Let X be a codimension k cycle in $\mathbb{P}^{n-1}$. The tropical Chow hypersurface of X is $\mathrm{Trop}(X) \boxplus (\mathcal{L}_{(k-1)})^{\mathrm{refl}}$.

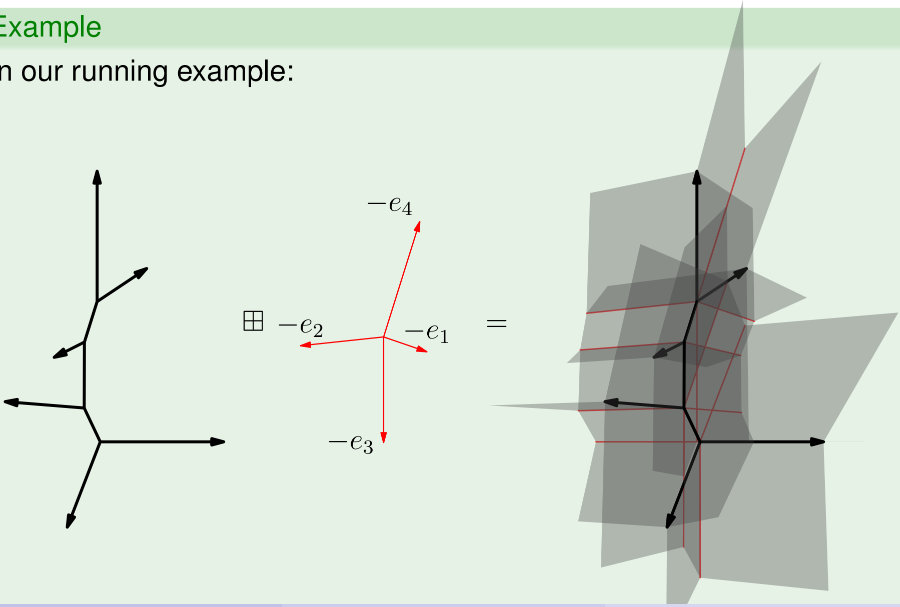
Definition

Define $ch : Z^k \to Z^1$ by $ch(C) = C \boxplus (\mathcal{L}_{(k-1)})^{\mathrm{refl}}$ for a tropical cycle C.

Computing a Chow hypersurface

Example

In our running example:



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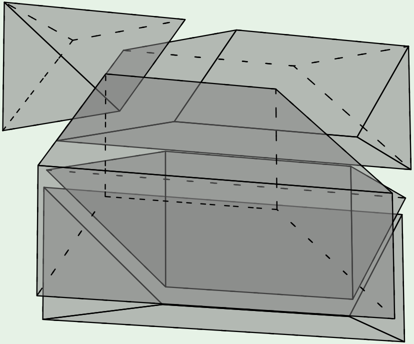

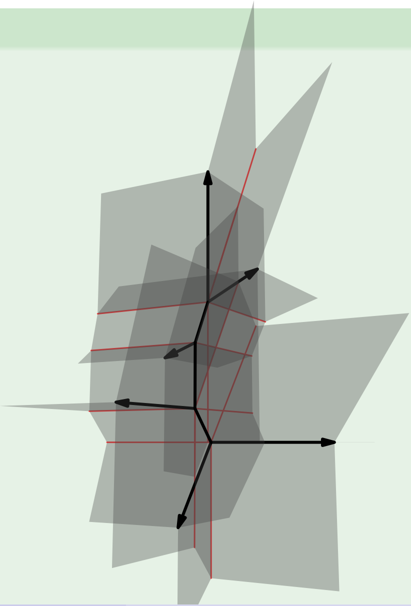

Aside: $\boxplus$ compared with intersection

In the exact sequence

$$0 \to \mathbb{R}^{n-1} \xrightarrow{\iota} \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \xrightarrow{\phi} \mathbb{R}^{n-1} \to 0$$

where ι is the inclusion along the diagonal and ϕ is subtraction,

$$C \cdot D = \iota^*(C \times D)$$
$$C \boxplus D^{\text{refl}} = \phi_*(C \times D).$$

If C and D have complimentary dimensions,

$$\#(C \cdot D) = \text{mult}(C \boxplus D^{\text{refl}}).$$

Degree

Definition

The degree of a tropical cycle C of codimension k is $\deg C := \#(C \cdot \mathcal{L}_{(k)})$.

Proposition

The degree of ch(C) *is* $\operatorname{codim} C \deg C$.

Definition

A tropical linear space is a tropical variety of degree 1.

Tropical linear spaces

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Others (e.g. Speyer) have taken tropical linear spaces in $\mathbb{TP}^{n-1}$ to be given by regular *matroid subdivisions*, described by Plücker vectors $(p_J : J \in \binom{[n]}{n-d})$.

Main theorem 2 (Mikhalkin–Sturmfels–Ziegler; F)

Every tropical linear space arises from a matroid subdivision. (That is, these definitions are equivalent.)

Matroid subdivision $\Rightarrow$ linear space is known:

$$(p_J) \quad \mapsto \quad \bigcap_{|J|=n-d+1} \operatorname{Trop} V\left(\bigoplus_{j\in K} a_{K\setminus j} \odot x_j\right)$$

This is the intersection of several hyperplanes, hence degree 1.

Sketch of proof: linear space $\Rightarrow$ matroid subdivision

Let C be a tropical linear space. We will construct the polytope subdivision Σ normal to $ch(C)$.
(Thus if $C = \mathrm{Trop}(X)$, we will construct $\mathrm{Chow}'(X)$. Good.)

Using relationships between $\boxplus$ and $\cdot$, show that Σ has $\{0, 1\}$-vector vertices and edge directions $e_i - e_j$. Thus Σ is a matroid polytope subdivision [Gelfand-Goresky-MacPherson-Serganova].

Why is Σ the right subdivision? We should be able to recover C from Σ by taking the normals to the *loop-free* faces [Ardila-Klivans].

Assume C has no lineality. Then:
normal to a loop-free face in $\Sigma \Leftrightarrow$ contains no ray in a direction $-e_i$;
C contains no rays in directions $-e_i$;
every ray of $(\mathcal{L}_{(k)})^{\mathrm{refl}}$ is in a direction $-e_i$.

The kernel of *ch*

$ch : Z^k \to Z^1$, $C \mapsto C \boxplus (\mathcal{L}_{(k)})^{\mathrm{refl}}$ is a linear map.
In each module Z^k of tropical cycles lies a pointed cone of varieties Z^k_{eff}, and we have $ch(Z^k_{\mathrm{eff}}) \subseteq Z^1_{\mathrm{eff}}$.

Fact

ch is not injective. Thus, Chow polytope subdivisions do not determine tropical varieties, in general.

Question 3

Describe the kernel of *ch*, and the fibers of its restriction to varieties.

Perhaps easier with fixed complexes, $ch : Z^k(\Sigma) \to Z^1(\Sigma')$.

Conjecture

ch is injective for curves.

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Some elements of ker *ch*: what's the fan?

Let $\mathcal{F}_n \subseteq \mathbb{R}^{n-1}$ be the normal fan of the *permutohedron*,
i.e. the fan of the type *A* reflection arrangement, the *braid arrangement*,
i.e. the common refinement of all normal fans of matroid polytopes.

The ray generators of $\mathcal{F}_n$ are $e_J = \sum_{j \in J} e_j$ for all $J \subsetneq [n], J \neq \emptyset$.
Its cones are generated by chains $\{e_{J_1}, \dots, e_{J_k} : J_1 \subseteq \cdots \subseteq J_k\}$.

The ring $Z^*(\mathcal{F}_n)$ is the cohomology ring of a generic torus orbit in the flag variety.

$\dim Z^*(\mathcal{F}_n) = n!$, and $\dim Z^k(\mathcal{F}_n)$ is the *Eulerian number* $E(n, k)$, i.e. the number of permutations of $[n]$ with *k descents*.

$n \setminus k$	0	1	2	3	4	5
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
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Tropical varieties with the same Chow polytope

For any cone $\sigma = \mathbb{R}_{\geq 0}\{e_{J_1}, \ldots, e_{J_k}\}$ of $\mathcal{F}_n$ and ${\sigma_{J'}}^{\mathrm{refl}} = \mathbb{R}_{\geq 0}\{-e_j : j \in J'\}$, the sum $\sigma \boxplus {\sigma_{J'}}^{\mathrm{refl}}$ is again a union of cones of $\mathcal{F}_n$.

So $ch(Z^k(\mathcal{F}_n)) \subseteq Z^1(\mathcal{F}_n)$. But $\dim Z^k(\mathcal{F}_n) > \dim Z^1(\mathcal{F}_n)$ for $1 < k < n-2$.

Example

For $(n, k) = (5, 2)$, $66 > 26$ and the kernel is 40-dimensional. Two tropical varieties in $\mathbb{R}^4$ of dim 2 with equal Chow polytope are

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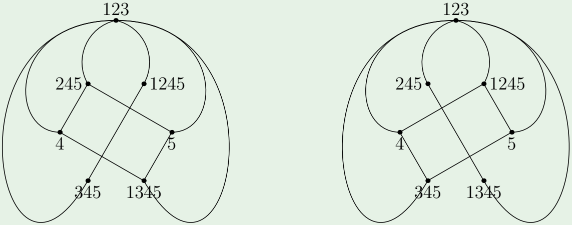

Take-home message

Tropical varieties are "dual" to their Chow subdivisions.

Trop var $\rightsquigarrow$ Chow subdiv has a nice combinatorial rule, in terms of stable Minkowski sum of tropical cycles.

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