BOIJ-S"ODERBERG EXPANSIONS OF MATROID STANLEY-REISNER RINGS

ALEX FINK

This note records a proof of Proposition 0.1 below, on a decomposition of matroid Stanley-Reisner rings into pure Boij-S"oderberg tables. We take the fundamental pure tables to be the vectors \( \pi_d \in \mathbb{Q}^Z \) indexed by sequences of positive integers \( d = (d_0, \ldots, d_c) \), such that the only nonzero components of \( \pi_d \) are
\[
(\pi_d)_{d_i} = \frac{(-1)^i}{\prod_{j \neq i} (d_j - d_i)}.
\]
We will always have \( d_0 = 0 \). We also write \( \{e_{ij}\} \) for the standard basis for the space \( \mathbb{Q}^Z \) of Betti tables.

Let \( S = k[x_1, \ldots, x_n] \). If \( \Delta \) is a simplicial complex on \( [n] = \{1, \ldots, n\} \), then \( I_{\Delta} \subseteq S \) will denote its Stanley-Reisner ideal. Matroids on the ground set \( [n] \) are interpreted as certain simplicial complexes on the vertices \( [n] \), whose faces are the independent sets: thus the rank of \( M \) is its dimension plus one. We use matroidal notation for operations on these complexes: for instance we denote restriction of the complex \( \Delta \) to a set \( A \) by \( \Delta|_A \).

For concision, let \( C(M) \) be the set of maximal chains of flats of a matroid \( M \). If the ground set of \( M \) is \( [n] \), this is the set of tuples \( F = (F_0, \ldots, F_{rk_M}) \) in which
\[
\emptyset = F_0 \subsetneq \cdots \subsetneq F_{rk_M} = [n]
\]
are all flats.

Proposition 0.1. If \( M \) is a matroid on \( [n] \) of rank \( r \) with no coloops, then the Betti table of the Stanley-Reisner ring \( S/I_M \) is given by
\[
(0.1) \quad \beta(S/I_M) = \sum_{F \in C(M)} \left( \prod_{i=1}^{n-r} |F_i| - |F_{i-1}| \right) \cdot \pi_{n-|F_{n-r}|, \ldots, n-|F_0|}.
\]

Proof. We will use Hochster’s formula [1, Corollary 5.12], in the following form:
\[
\beta_{ij}(S/I_M) = \sum_{A \subseteq [n]} \dim \tilde{H}^{j-i-1}(M|A, k).
\]
These restrictions \( M|A \) of the matroid \( M \) are themselves matroids and are therefore Cohen-Macaulay, and so \( \dim \tilde{H}^{j-i-1}(M|A, k) \) is only nonzero if \( j - i - 1 \) is equal to the dimension of \( M|A \), i.e. if \( j - i = rk_M(A) \). The dimension of the top-dimensional homology of \( M|A \) is the Tutte evaluation \( T_{M|A}(0, 1) \). So the above sum may be recast
\[
\beta(S/I_M) = \sum_{A \subseteq [n]} T_{M|A}(0, 1) e_{|A|-rk_M(A), |A|}.
\]
Changing to the dual matroid, and writing \( F = [n] \setminus A \), this is

\[
0.2 \quad \beta(S/I_M) = \sum_{F \subseteq [n]} T_{M^*/F}(1, 0) e_{n-r-\text{rk}_{M^*}(F), n-|F|}.
\]

Let us now turn to the right side of (0.1). Expanding the definition of the \( \pi_d \), this is

\[
\sum_{F \text{ a flat}} e_{n-\text{rk}_{M^*}(F), n-|F|} \left( \sum_{G \in \mathcal{C}(M^*/F)} \prod_{j=1}^{\text{rk}_{M^*}/F} \frac{|G_j| - |G_{j-1}|}{|F| - |G_{j-1}|} \right) \left( \sum_{H \in \mathcal{C}(M^*/F)} \prod_{j=1}^{\text{rk}_{M^*}/F} \frac{|H_j| - |H_{j-1}|}{|H_j|} \right).
\]

We recast this as a sum over the various flats \( F := F_i \) of \( M^* \) that occur in the chains \( F \), breaking up the remaining summation into the subchain of \( F \) before the \( i \)th position and the subchain after. Note that \( i = \text{rk}_{M^*}(F) \). What results is

\[
\sum_{F \text{ a flat}} e_{n-\text{rk}_{M^*}(F), n-|F|} \left( \sum_{G \in \mathcal{C}(M^*/F)} \prod_{j=1}^{\text{rk}_{M^*}/F} \frac{|G_j| - |G_{j-1}|}{|F| - |G_{j-1}|} \right) \left( \sum_{H \in \mathcal{C}(M^*/F)} \prod_{j=1}^{\text{rk}_{M^*}/F} \frac{|H_j| - |H_{j-1}|}{|H_j|} \right).
\]

We now compare this sum to (0.2). First of all, the terms of (0.2) for which \( F \) is not a flat of \( M^* \) make no contribution, as then \( M^*/F \) contains a loop, making \( T_{M^*/F}(1, 0) \) equal to 0. We are thus done in view of the equations in Lemma 0.2 for the two parenthesized factors. \( M^*/F \) is loopfree because \( M^* \) is; \( M^*/F \) is because \( F \) is a flat.)

\[\square\]

**Lemma 0.2.** Let \( M \) be a matroid on ground set \([n]\) with no loops. Then

\[
(a) \quad \sum_{F \in \mathcal{C}(M)} \prod_{j=1}^{\text{rk}_{M}} \frac{|F_j| - |F_{j-1}|}{n - |F_{j-1}|} = 1.
\]

\[
(b) \quad \sum_{F \in \mathcal{C}(M)} \prod_{j=1}^{\text{rk}_{M}} \frac{|F_j| - |F_{j-1}|}{|F_j|} = T_M(1, 0).
\]

**Proof.** In both cases the proof will be inductive on the rank of \( M \), by taking subchains of length one less and passing to an appropriate minor of \( M \). The rank 0 base cases are trivial.

For (a), we extract the \( j = 1 \) term of the product, giving

\[
\sum_{F \text{ a rank 1 flat}} \left( \frac{|F|}{n} \sum_{G \in \mathcal{C}(M/F)} \prod_{i=1}^{\text{rk}_{M}/F} \frac{|G_i| - |G_{i-1}|}{n - |F_i| - |G_{i-1}|} \right)
\]

by induction. Since the rank 1 flats partition \([n]\), the sum above equals 1 as desired.

For (b), we begin by noting that \( T_M(1, 0) \) is the Möbius function evaluation \((-1)^{\text{rk}_{M}}\mu(\emptyset, [n])\) in the lattice of flats of \( M \). (This follows from the Crosscut Theorem [2, Corollary 3.9.4], since by the corank-nullity expansion of Tutte, \( T_M(1, 0) \) counts spanning sets of \( M \) with alternating sign.)
Using the induction, we extract the $j = \text{rk} M$ term of the product and have

$$\sum_{F \text{ a hyperplane}} \left( \frac{n - |F|}{n} \sum_{G \in \mathcal{C}(M|F)} \left( \prod_{j=1}^{\text{rk} M - 1} \frac{|G_j| - |G_{j-1}|}{|G_j|} \right) \right)$$

$$= \sum_{F \text{ a hyperplane}} \frac{n - |F|}{n} \cdot (-1)^{\text{rk} M - 1} \mu(\emptyset, F)$$

$$= \frac{1}{n} \sum_{a \in [n]} \sum_{F \not\ni a \text{ a hyperplane}} (-1)^{\text{rk} M - 1} \mu(\emptyset, F)$$

$$= \frac{1}{n} \sum_{a \in [n]} (-1)^{\text{rk} M} \mu(\emptyset, [n])$$

$$= (-1)^{\text{rk} M} \mu(\emptyset, [n]),$$

where the second-last equality is Weisner's theorem [2, Corollary 3.9.3].

[[Eliminate the no-coloops restriction. Is this better framed in terms of the cover ideal, and does it then go through for non-matroids? Are there connections between the product on Boij-Söderberg tables and my Hopf structures with Derksen?]]

**References**
