6 Linear Programming Duality

Consider the linear program (1.2) and introduce slack variables z to turn it into

$$\min\{\, c^T x : Ax - z = b, x, z \geqslant 0 \,\}.$$

We have $X = \{(x, z) : x \geqslant 0, z \geqslant 0\} \subseteq \mathbb{R}^{m+n}$. The Lagrangian is given by

$$L((x,z),\lambda) = c^T x - \lambda^T(Ax - z - b) = (c^T - \lambda^T A)x + \lambda^T z + \lambda^T b$$

and has a finite minimum over X if and only if

$$\lambda \in Y = \{\, \mu : c^T - \mu^T A \geqslant 0, \mu \geqslant 0 \,\}.$$

For $\lambda \in Y$, the minimum of $L((x,z),\lambda)$ is attained when both $(c^T - \lambda^T A)x = 0$ and $\lambda^T z = 0$, and thus

$$g(\lambda) = \inf_{(x,z)\in X} L((x,z),\lambda) = \lambda^T b.$$

We obtain the dual

$$\max\{\, b^T\lambda : A^T\lambda \leqslant c, \lambda \geqslant 0 \,\}. \tag{6.1}$$

The dual of (1.3) can be determined analogously as

$$\max\{\, b^T\lambda : A^T\lambda \leqslant c \,\}.$$

The dual is itself a linear program, and its dual is in fact equivalent to the primal.

Theorem 6.1. *In the case of linear programming, the dual of the dual is the primal.*

Proof. The dual can be written equivalently as

$$\min\{\, -b^T\lambda : -A^T\lambda \geqslant -c, \lambda \geqslant 0 \,\}.$$

This problem has the same form as the primal (1.2), with $-b$ taking the role of c, $-c$ taking the role of b, and $-A^T$ the role of A. Taking the dual again we thus return to the original problem. □

6.1 The Relationship between Primal and Dual

Example 6.2. Consider the following pair of a primal and dual LP, with slack variables z_1 and z_2 for the primal and μ_1 and μ_2 for the dual.

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 + z_1 = 4 \\ & 2x_1 + 3x_2 + z_2 = 6 \\ & x_1, x_2, z_1, z_2 \geqslant 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & 4\lambda_1 + 6\lambda_2 \\ \text{subject to} & 2\lambda_1 + 2\lambda_2 - \mu_1 = 3 \\ & \lambda_1 + 3\lambda_2 - \mu_2 = 2 \\ & \lambda_1, \lambda_2, \mu_1, \mu_2 \geqslant 0 \end{array}$$

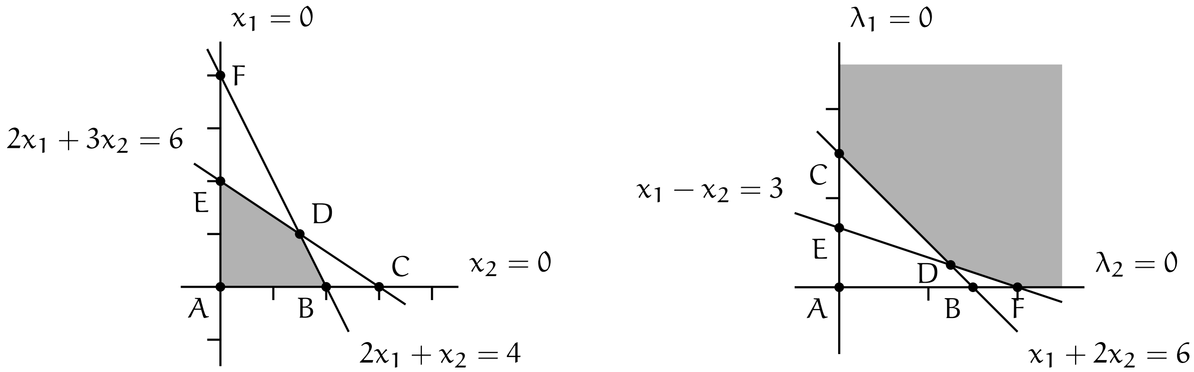

Figure 6.1: Geometric interpretation of primal and dual linear programs in Example 6.2

To see that these LPs are indeed dual to each other, observe that the primal has the form (1.2), and the dual the form (6.1), with

$$c = -\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad A = -\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}, \quad b = -\begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$

As before, we can compute all basic solutions of the primal by setting any set of $n - m = 2$ variables to zero in turn, and solving for the values of the remaining $m = 2$ variables. Given a particular basic solution of the primal, the corresponding dual solution can be found using the complementary slackness conditions $\lambda_1 z_1 = 0 = \lambda_2 z_2$ and $\mu_1 x_1 = 0 = \mu_2 x_2$. These conditions identify, for each non-zero variable of the primal, a dual variable whose value has to be equal to zero. By solving for the remaining variables, we obtain a solution for the dual, which is in fact a basic solution. Repeating this procedure for every basic solution of the primal, we obtain the following pairs of basic solutions of the primal and dual:

	x_1	x_2	z_1	z_2	$f(x)$	λ_1	λ_2	μ_1	μ_2	$g(\lambda)$
A	0	0	4	6	0	0	0	-3	-2	0
B	2	0	0	2	6	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	6
C	3	0	-2	0	9	0	$\frac{3}{2}$	0	$\frac{5}{2}$	9
D	$\frac{3}{2}$	1	0	0	$\frac{13}{2}$	$\frac{5}{4}$	$\frac{1}{4}$	0	0	$\frac{13}{2}$
E	0	2	2	0	4	0	$\frac{2}{3}$	$-\frac{5}{3}$	0	4
F	0	4	0	-6	8	2	0	1	0	8

Labels A through F refer to Figure 6.2, which illustrates the feasible regions of the primal and the dual. Observe that there is only one pair such that both the primal and the dual solution are feasible, the one labeled D, and that these solutions are optimal.

6.2 Necessary and Sufficient Conditions for Optimality

In the above example, primal feasibility, dual feasibility, and complementary slackness together imply optimality. It turns out that this is true in general, and the condition is in fact both necessary and sufficient for optimality.

THEOREM 6.3. *Let x and λ be feasible solutions for the primal* (1.2) *and the dual* (6.1)*, respectively. Then x and λ are optimal if and only if they satisfy complementary slackness, i.e., if*

$$(c^T - \lambda^T A)x = 0 \quad \textit{and} \quad \lambda^T(Ax - b) = 0.$$

Proof. If x and λ are optimal, then

$$\begin{aligned} c^T x &= \lambda^T b \\ &= \inf_{x' \in X} \left(c^T x' - \lambda^T(Ax' - b) \right) \\ &\leqslant c^T x - \lambda^T(Ax - b) \\ &\leqslant c^T x. \end{aligned}$$

Since the first and last term are the same, the two inequalities must hold with equality. Therefore, $\lambda^T b = c^T x - \lambda^T(Ax - b) = (c^T - \lambda^T A)x + \lambda^T b$, and thus $(c^T - \lambda^T A)x = 0$. Furthermore, $c^T x - \lambda^T(Ax - b) = c^T x$, and thus $\lambda^T(Ax - b) = 0$.

If on the other hand $(c^T - \lambda^T A)x = 0$ and $\lambda^T(Ax - b) = 0$, then

$$c^T x = c^T x - \lambda^T(Ax - b) = (c^T - \lambda^T A)x + \lambda^T b = \lambda^T b,$$

and by weak duality x and λ must be optimal. □

While the result has been formulated here for the primal LP in general form and the corresponding dual, it is true, with the appropriate complementary slackness conditions, for any pair of a primal and dual LP.