Optimization **Example Sheet 1**
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1 Minimize the following functions subject to $x \geqslant 0$:

(a) $3x$ (b) $x^2 - 2x + 3$ (c) $x^2 + 2x + 3$.

For each of the following functions specify the set Y of values of λ for which the function has a finite minimum in the region specified, and for each $\lambda \in Y$ find the minimum and all optimal solutions.

(d) λx subject to $x \geqslant 0$ (e) λx subject to $x \in \mathbb{R}$ (f) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \in \mathbb{R}$
(g) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \geqslant 0$ (h) $(\lambda_1 - \lambda_2)x$ subject to $0 \leqslant x \leqslant M$.

2 Show how to

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} \frac{1}{(a_i + x_i)} & \\ \text{subject to} & \sum_{i=1}^{n} x_i = b & \\ & x_i \geqslant 0 & \text{for } i = 1, \dots, n. \end{array}$$

where $a_i > 0$ for $i = 1, \dots, n$ and $b > 0$.

3 Maximize $n_1 \log p_1 + \cdots + n_k \log p_k$ subject to $p_1 + \cdots + p_k = 1$, $p_1, \dots, p_k > 0$, where $n_1, \dots, n_k > 0$ are given. (Note that the optimal solution is the maximum likelihood estimator for the multinomial distribution, $p(n_1, \dots, n_k) = \binom{n}{n_1, \dots, n_k} p_1^{n_1} \dots p_k^{n_k}$.)

4 A probability vector is a vector $p = (p_1, \dots, p_n)^T$ such that $p_i \geqslant 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$. The entropy $H(p)$ of probability vector p is defined as

$$H(p) = -\sum_{i=1}^{n} p_i \log p_i,$$

where $0 \log 0 = 0$ by convention. What is the largest and smallest entropy of any probability vector?

5 Maximize $2 \tan^{-1} x_1 + x_2$ subject to $x_1 + x_2 \leqslant b_1$, $-\log x_2 \leqslant b_2$, and $x_1, x_2 \geqslant 0$, where b_1 and b_2 are constants such that $b_1 - e^{-b_2} \geqslant 0$. (Think carefully about the two cases in which the Lagrange multiplier for the second constraint is equal to 0 or greater than 0.)

6 Consider the following optimization problems:

$$\begin{aligned} &\text{maximize } c^T x \text{ subject to } Ax = b, \\ &\text{maximize } c^T x \text{ subject to } Ax \geqslant b, \\ &\text{minimize } c^T x \text{ subject to } Ax = b,\ x \geqslant 0, \\ &\text{minimize } c^T x \text{ subject to } Ax \leqslant b,\ x \leqslant 0. \end{aligned}$$

For each of these problems,

(i) determine the Lagrangian and the set of Lagrange multipliers for which the Lagrangian has a finite minimum subject to a possible regional constraint, and write down the dual problem;

(ii) determine the necessary and sufficient conditions for optimality; and

(iii) verify that the dual of the dual is the primal.

7 Consider the problem to

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leqslant 4 \\ & x_1 + 2x_2 \leqslant 4 \\ & x_1 - x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

(a) Solve the problem graphically in the plane.

(b) Introduce slack variables x_3, x_4, and x_5 and write the problem in equality form. How many basic solutions are there? Determine the value of $x = (x_1, \dots, x_5)^T$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all basic solutions non-degenerate?

(c) Write down the dual problem in equality form using slack variables λ_4 and λ_5, and determine the value of $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ and of the objective function at each of the basic solutions of the dual. Which of these basic solutions are feasible?

(d) Write down the complementary slackness conditions for the problem, and show that for each basic solution of the primal there is exactly one basic solution of the dual such that the two have the same value and satisfy complementary slackness. How many of these pairs are feasible for both primal and dual?

(e) Solve the problem using the simplex method. Start from the basic feasible solution where $x_1 = x_2 = 0$, and try both choices for a variable to enter into the basis. How are the entries in the last row of the various tableaus related to the appropriate basic solutions of the dual?

8 Consider the problem to

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \leqslant 2 \\ & x_1 + 2x_2 + 3x_3 \leqslant 5 \\ & 2x_1 + 2x_2 + x_3 \leqslant 6 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

(a) Use the simplex method to solve this problem.

(b) Explain why each row of the final tableau must be the sum of scalar multiples of the rows of the initial tableau, and how the multipliers can be determined from the final tableau.

(c) Let $P(\epsilon)$ denote the linear program obtained by replacing the vector $b = (2, 5, 6)^T$ by the perturbed vector $b(\epsilon) = (2 + \epsilon_1, 5 + \epsilon_2, 6 + \epsilon_3)^T$. Derive a formula that describes the optimal value of $P(\epsilon)$ in terms of ϵ when the entries of ϵ are small. For which values of ϵ_1 does the formula hold if $\epsilon_2 = \epsilon_3 = 0$?

9 Use the simplex method to

$$\begin{aligned} \text{maximize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 - 2x_2 \leqslant 4 \\ & -x_1 + x_2 \leqslant 3 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Explain what happens with the help of a diagram.

10 Use the two-phase simplex method to

$$\begin{aligned} \text{maximize} \quad & -2x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 - 2x_2 \leqslant 1 \\ & 5x_1 + 3x_2 \geqslant 3 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Confirm that the maximum occurs at $x_1 = 9/16$ and $x_2 = 1/16$. Note that the first pivot column can be chosen such that phase I ends after only one round.

11 Use the two-phase simplex method to

$$\begin{aligned} \text{minimize} \quad & 13x_1 + 5x_2 - 12x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 \leqslant 5 \\ & 3x_1 + 3x_2 + x_3 \geqslant 7 \\ & x_1 + 5x_2 + 4x_3 = 10 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

12 Consider the problem to

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geqslant 4 \\ & 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geqslant 3 \\ & x_1, x_2, x_3, x_4, x_5 \geqslant 0. \end{aligned}$$

Write down the dual problem and solve it graphically. Then deduce the optimal solution of the primal.