19 Cooperative Games

So far we have considered non-cooperative games, in which each players acts on its own. The theory of cooperative games, on the other hand, studies which coalitions of players are likely to form in a given situation. It takes the payoff achievable by any coalition as given, but requires that coalitions can distribute these payoffs among their members in such a way that the members are satisfied. We consider games with transferable payoff, where the payoff obtained by a coalition can be distributed in an arbitrary way among its members, and restrict our attention to the grand coalition N consisting of all players.

Formally, a *coalitional game* is given by a set $N = \{1, \dots, n\}$ of *players*, and a *characteristic function* $v : 2^N \to \mathbb{R}$ that maps each coalition of players to its *value*, the joint payoff the coalition can obtain by working together. For a given game (N, v), a vector $x \in \mathbb{R}^n$ of payoffs is said to satisfy *(economic) efficiency* if $\sum_{i \in N} x_i = v(N)$ and *individual rationality* if $x_i \geqslant v(\{i\})$ for $i = 1, \dots, n$. The first condition intuitively ensures that no payoff is wasted, while the second condition ensures that each player obtains at least the same payoff it would be able to obtain on its own. A payoff vector that is both efficient and individually rational is also called an *imputation*.

19.1 The Core

Efficiency and individual rationality may not be enough to guarantee a stable outcome. For any two imputations x and y, $\sum_{i \in N} x_i = \sum_{i \in N} y_i = v(N)$, so $y_i > x_i$ for some $i \in N$ implies that $y_j < x_j$ for some other $j \in N$. However, there could be some coalition $S \subseteq N$ such that $y_i > x_i$ for all $i \in S$. If in addition $\sum_{i \in S} y_i \leqslant v(S)$, the members of S could increase their respective payoffs by deviating from the grand coalition, forming the coalition S, and distributing the payoff thus obtained according to y. The core is the set of imputations that are stable against this kind of deviation. Formally, imputation x is in the core of game (N, v) if $\sum_{i \in S} x_i \geqslant v(S)$ for all $S \subseteq N$.

Consider a situation where $n \geqslant 2$ members of an expedition have discovered a treasure, and any pair of them can carry one piece of the treasure back home. This situation can be modeled by a coalitional game (N, v) where $N = \{1, \dots, n\}$ and $v(S) = |S|/2$ if $|S|$ is even and $v(S) = (|S| - 1)/2$ if $|S|$ is odd. The core then contains all imputations if $n = 2$, the single imputation $(1/2, \dots, 1/2)$ if $n \geqslant 4$ is even, and is empty if n is odd. The latter can for example be shown using a characterization of games with a non-empty core, which we discuss next.

Call a function $\lambda : 2^N \to [0, 1]$ *balanced* if for every player the weights of all coalitions containing that player sum to 1, i.e., if for all $i \in N$, $\sum_{S \subseteq N \setminus \{i\}} \lambda(S \cup \{i\}) = 1$. A game (N, v) is called *balanced* if for every balanced function λ, $\sum_{S \subseteq N} \lambda(S) v(S) \leqslant v(N)$. The intuition behind this definition is that each player allocates one unit of time among the coalitions it is a member of, and each coalition earns a fraction of its value proportional to the minimum amount of time devoted to it by any of its members. Balancedness of a collection of weights imposes a feasibility condition on players' allocations of time, and a game is balanced if there is no feasible allocation that yields more than $v(N)$.

THEOREM 19.1 (Bondareva 1963, Shapley 1967). *A game has a non-empty core if and only if it is balanced.*

Proof. The core of a game (N, v) is non-empty if and only if the linear program to

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in N} x_i \\ \text{subject to} \quad & \sum_{i \in S} x_i \geqslant v(S) \quad \text{for all } S \subseteq N \end{aligned}$$

has an optimal solution with value $v(N)$. This linear program has the following dual:

$$\begin{aligned} \text{maximize} \quad & \sum_{S \subseteq N} \lambda(S) v(S) \\ \text{subject to} \quad & \sum_{S \subseteq N, i \in S} \lambda(S) = 1 \quad \text{for all } i \in N \\ & \lambda(S) \geqslant 0 \quad \text{for all } S \subseteq N, \end{aligned}$$

where $\lambda : 2^N \to [0, 1]$. Note that λ is feasible for the dual if and only if it is a balanced function. Both primal and dual are feasible, so by strong duality their optimal objective values are the same. This means that the core is non-empty if and only if $\sum_{S \subseteq N} \lambda(S) v(S) \leqslant v(N)$ for every balanced function λ. □

To see that the core of our example game is empty if n is odd, define $\lambda : 2^N \to [0, 1]$ such that $\lambda(S) = 1/(n-1)$ if $|S| = 2$ and $\lambda(S) = 0$ otherwise. Then, for all $i \in N$, $\sum_{S \subseteq N \setminus \{i\}} \lambda(S \cup \{i\}) = 1$, because each player is contained in exactly $(n-1)$ sets of size 2. Moreover, $\sum_{S \subseteq N} \lambda(S) v(S) = n(n-1)/2 \cdot 1/(n-1) = n/2$, which is greater than $v(N)$ if n is odd.

19.2 The Nucleolus

In cases where the core is empty, one might consider weakening the requirement that no coalition should be able to gain, and instead look for an efficient payoff vector that minimizes the possible gain over all coalitions. This can intuitively be interpreted as minimizing players' incentive to deviate from the solution by forming another coalition, or as a natural notion of fairness when distributing the joint payoff $v(N)$ among the players.

To this end, define the *excess* $e(S, x)$ of coalition $S \subseteq N$ for payoff vector x as its gain from leaving the grand coalition, i.e., $e(S, x) = v(S) - \sum_{i \in S} x_i$. For a given vector

x, let $S_1^x, \dots, S_{2^n-1}^x$ be an ordering of the coalitions such that $e(S_k^x, x) \geqslant e(S_{k+1}^x, x)$ for $k = 1, \dots, 2^n - 2$, and let $E(x) \in \mathbb{R}^{2^n-1}$ be the vector given by $E_k(x) = e(S_k^x, x)$. We say that $E(x)$ is lexicographically smaller than $E(y)$ if there exists $i \in \{1, \dots, 2^n - 1\}$ such that $E_k(x) = E_k(y)$ for $k = 1, \dots, i-1$ and $E_i(x) < E_i(y)$. The *nucleolus* is then defined as the set of efficient payoff vectors x for which $E(x)$ is lexicographically minimal.

Observe that for each $k = 1, \dots, 2^n - 1$, E_k is a continuous function because

$$E_k(x) = \min_{\substack{\mathcal{T} \subseteq 2^N \\ |\mathcal{T}| = k-1}} \max_{S \in 2^N \setminus \mathcal{T}} e(S, x).$$

Since E_1 is continuous, $X_1 = \arg\min_{x \in X_0} E_1(x)$, where X_0 is the set of efficient payoff vectors, is non-empty and compact. By induction, the same holds for $X_k = \arg\min_{x \in X_{k-1}} E_k(x)$, $k \geqslant 2$. Since X_{2^n-1} is the nucleolus, the nucleolus is non-empty. It can in fact be shown that the nucleolus always contains exactly one element.

Theorem 19.2. *The nucleolus of any coalitional game is a singleton.*

Proof. Consider two vectors x and y in the nucleolus, and assume for contradiction that $x \neq y$. Observe that $E(x) = E(y)$, and let $S_1, \dots, S_{2^n-1}$ be an ordering of the coalitions such that $e(S_k, x) \geqslant e(S_{k+1}, x)$ for $k = 1, \dots, 2^n - 2$. Since $x \neq y$, there has to exist $\ell \in \{1, \dots, 2^n - 1\}$ such that $e(S_k, x) = e(S_k, y)$ for $k = 1, \dots, \ell - 1$ and $e(S_\ell, x) \neq e(S_\ell, y)$. In fact $e(S_\ell, x) > e(S_\ell, y)$, because $e(S_\ell, x) < e(S_\ell, y)$ would imply that $E(x)$ is lexicographically smaller than $E(y)$. Moreover, for $k = \ell + 1, \dots, 2^n - 1$, $E_k(x) \leqslant E_\ell(x)$ and $e(S_k, y) \leqslant E_\ell(x)$. The latter inequality holds because $e(S_k, y) = E_k(y)$ for $k = 1, \dots, \ell - 1$, so it must be the case that $e(S_k, y) \leqslant E_\ell(y) = E_\ell(x)$ for $k = \ell + 1, \dots, 2^n - 1$.

Now consider the vector $z = (x + y)/2$, and observe that $e(S_k, z) = (e(S_k, x) + e(S_k, y))/2$ for $k = 1, \dots, 2^n - 1$. Thus $E_k(z) = E_k(x)$ for $k = 1, \dots, \ell - 1$, $E_\ell(z) < E_\ell(x)$, and $E_k(z) < E_\ell(x)$ for $k = \ell + 1, \dots, 2^n - 1$. This means that $E(z)$ is lexicographically smaller than $E(x)$, contradicting the assumption that x is in the nucleolus. □

The set of efficient payoff vectors that minimize maximum excess is the set of optimal solutions of the following linear program:

$$\begin{aligned} \text{minimize} \quad & \epsilon \\ \text{subject to} \quad & \textstyle\sum_{i \in S} x_i \geqslant v(S) - \epsilon \quad \text{for all } S \subset N \\ & \textstyle\sum_{i \in N} x_i = v(N). \end{aligned} \tag{P_1}$$

The set of feasible solutions of (P_1) for a fixed value of ϵ is also called the ϵ-*core*, and the set of optimal solutions of (P_1) the *least core*. The core is non-empty if and only if the optimal objective value of (P_1) is non-positive.

Now let $\mathcal{S}_1 \subseteq 2^N \setminus \{N\}$ be the set of coalitions whose constraint holds with equality in *every* optimal solution of (P_1). Clearly, if x is in the nucleolus, then $e(S, x) = E_1(x)$ for all $S \in \mathcal{S}_1$. Given that $\mathcal{S}_1$ is non-empty, we can thus write down a new linear program that fixes the excess of the coalitions in $\mathcal{S}_1$ to $E_1(x)$ and minimizes the next smaller excess. By repeating this procedure we obtain a sequence of linear programs $P_2, P_3, \ldots$ defined recursively as follows:

$$\begin{array}{lll} \text{minimize} & \epsilon & \\ \text{subject to} & \sum_{i \in S} x_i = v(S) - \epsilon_1 & \text{for all } S \in \mathcal{S}_1 \\ & \vdots & \\ & \sum_{i \in S} x_i = v(S) - \epsilon_{i-1} & \text{for all } S \in \mathcal{S}_{i-1} \\ & \sum_{i \in S} x_i \geqslant v(S) - \epsilon & \text{for all } S \in \bar{\mathcal{S}}_{i-1} \\ & \sum_{i \in N} x_i = v(N), & \end{array} \tag{P_i}$$

where for $j = 1, \ldots, i-1$, ϵ_j is the optimal objective value of (P_j), $\mathcal{S}_j \subseteq \bar{\mathcal{S}}_{j-1}$ is the set of coalitions with an inequality constraint in (P_j) that holds with equality in every optimal solution, and $\bar{\mathcal{S}}_j = (2^N \setminus \{N\}) \setminus \cup_{\ell=1}^{j} \mathcal{S}_\ell$.

The following result establishes that there always exists an inequality constraint that can be tightened to an equality, which guarantees that we obtain the nucleolus after a finite number of iterations. In fact, n iterations can be shown to suffice if in addition to the constraints that are tight in every optimal solution we also fix those that are linearly dependent on constraints that have already been fixed.

LEMMA 19.3. *If $\bar{\mathcal{S}}_{i-1} \neq \emptyset$, then $\mathcal{S}_i \neq \emptyset$.*

Proof. Let $\bar{\mathcal{S}}_{i-1} = \{S_1, \ldots, S_m\}$. If $m = 0$ or if the constraint for some $S \in \bar{\mathcal{S}}_{i-1}$ holds with equality in every optimal solution of (P_i) we are done. Otherwise, for $j = 1, \ldots, m$, there exists an optimal solution x^j of (P_i) such that $e(S, x^j) < \epsilon_i$. By convexity of the set of optimal solutions, $\tilde{x} = 1/m \sum_{i=1}^{m} x^j$ is optimal for (P_i), which means that there has to be a coalition $\tilde{S} \in \bar{\mathcal{S}}_{i-1}$ such that $e(\tilde{S}, \tilde{x}) = \epsilon_i$. Thus,

$$e(\tilde{S}, \tilde{x}) = \frac{1}{m} \sum_{j=1}^{m} e(\tilde{S}, x^j) \leqslant \epsilon_i,$$

where the equality holds by convexity of e and the inequality because for $j = 1, \ldots, m$, x^j is optimal for (P_i) and thus $e(\tilde{S}, x^j) \leqslant \epsilon_i$. If the constraint for $\tilde{S}$ had slack for some optimal solution of (P_i), i.e., if $\tilde{S} = S_j$ for some $j = 1, \ldots, m$, then this inequality would be strict. Since it is not, the constraint for $\tilde{S}$ must hold with equality in every optimal solution of (P_i). □

Consider for example a game with $N = \{1, 2, 3\}$ and characteristic function v given by

$$\begin{array}{c} v(\{1\}) = 1 \quad v(\{2\}) = 2 \quad v(\{3\}) = 1 \\ v(\{1,2\}) = 2 \quad v(\{1,3\}) = 3 \quad v(\{2,3\}) = 5 \quad v(\{1,2,3\}) = 4. \end{array} \tag{19.1}$$

To find the nucleolus of this game, we first

$$\begin{aligned} \text{minimize} \quad & \epsilon \\ \text{subject to} \quad & x_1 \geqslant 1 - \epsilon, \quad x_2 \geqslant 2 - \epsilon, \quad x_3 \geqslant 1 - \epsilon \\ & x_1 + x_2 \geqslant 2 - \epsilon, \quad x_1 + x_3 \geqslant 3 - \epsilon \\ & x_2 + x_3 \geqslant 5 - \epsilon, \quad x_1 + x_2 + x_3 = 4. \end{aligned}$$

It is easily verified that $\epsilon = 1$ and $x = (0, 1, 3)$ is a feasible solution. By adding the constraints for $\{1\}$ and $\{2, 3\}$ and subtracting the constraint for $\{1, 2, 3\}$ we see that $\epsilon \geqslant 1$, so the solution must be optimal. For $\epsilon = 1$, the constraints for $\{1\}$ and $\{2, 3\}$ have to hold with equality in every optimal solution, and the constraints for $\{3\}$, $\{1, 2\}$, and $\{1, 2, 3\}$ become redundant. Thus $x_1 = 0$, $x_2 \geqslant 1$, $x_3 \geqslant 2$, and $x_2 + x_3 = 4$, and the least core is $\{(0, x_2, 4 - x_2) : x_2 \in [1, 2]\}$. We now

$$\begin{aligned} \text{minimize} \quad & \epsilon \\ \text{subject to} \quad & x_1 = 0, \quad x_2 \geqslant 2 - \epsilon \\ & x_3 \geqslant 3 - \epsilon, \quad x_2 + x_3 = 4 \end{aligned}$$

and obtain a unique optimal solution where $\epsilon = 1/2$ and $x = (0, 3/2, 5/2)$, which gives us the unique element in the nucleolus.

19.3 The Shapley Value

A different notion of fairness in distributing the joint payoff of a coalition among its members was proposed by Shapley, starting from a set of axioms. Call player $i \in N$ a *dummy* if its contribution to every coalition is exactly its value, i.e., if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. Call two players $i, j \in N$ *interchangeable* if they contribute the same to every coalition, i.e., if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Let a *solution* be a function $\phi : \mathbb{R}^{2^n} \to \mathbb{R}^n$ that maps every characteristic function v to an efficient payoff vector $\phi(v)$. Solution ϕ is said to satisfy

- *dummies* if $\phi_i(v) = v(\{i\})$ whenever i is a dummy;
- *symmetry* if $\phi_i(v) = \phi_j(v)$ whenever i and j are interchangeable; and
- additivity if $\phi(v + w) = \phi(v) + \phi(w)$.

It turns out that there is a unique solution satisfying these axioms.

THEOREM 19.4. *The* Shapley value, *given by*

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \big(v(S \cup \{i\}) - v(S)\big),$$

is the unique solution that satisfies dummies, symmetry, and additivity.