Payment Rules for Combinatorial Auctions via Structural Support Vector Machines

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Combinatorial Auctions

- $n$ agents
- $m$ items
- Bundles $Y = \{0, 1\}^m$
- Valuation profiles $X = \mathbb{R}^{2^m \times n}$
- Allocation rule $g_i : X \rightarrow Y$
- Payment rule $t_i : X \times Y \rightarrow \mathbb{R}$

- Optimal allocation: maximize $\sum_i x_i[y_i]$ such that $y_i \cap y_j = \emptyset$
- Strategyproofness:
  \[
  x_i[g_i(x)] - t_i(x, g_i(x)) \geq x_i[g_i(x'_i, x_{-i})] - t_i(x'_i, x_{-i}, g_i(x'_i, x_{-1}))
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Problem Statement

- Elicitation of valuations and computation of optimal allocation are costly, often prohibitively so
- Canonical strategyproof mechanism: VCG
  - depends on ability to find efficient allocation
  - other problems: collusion, small or non-monotonic revenue
- Alternative solutions hard to come by
- Our approach: take allocation rule $g$ as given, use to generate input for a learning algorithm
- Implicitly learns payment rule $t$ that makes $g$ “maximally incentive compatible” (we will see in what sense)
Outline

Combinatorial Auctions and Margin-Based Learning

Learning a Payment Rule

Summary and Open Problems
Learning What We Already Know

- By symmetry concentrate on agent 1, consider $g = g_1$ and $t = t_1$
- Assume $g$ is given, as well as a distribution $P(X)$ on $X$
- Together they induce a distribution $P(X, Y)$ on $X \times Y$
- Sample set of training examples from $P(X, Y)$ and learn an allocation function $h : X \rightarrow Y$
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- Assume \( g \) is given, as well as a distribution \( P(X) \) on \( X \)
- Together they induce a distribution \( P(X, Y) \) on \( X \times Y \)
- Sample set of training examples from \( P(X, Y) \) and learn an allocation function \( h : X \to Y \)
- We know \( g \), so we are not actually interested in \( h \)
- Rather: employ a margin-based learning method, infer \( t \) from the margin
Learning How to Allocate

- Single-item case corresponds to an ordinary binary classifier: allocate the item or not

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![Diagram of binary classifier](image)
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![Diagram showing single-item allocation decision](image)
Learning How to Allocate

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- In general: one class for each bundle that could be allocated
- Learn a discriminant function $f : X \times Y \rightarrow \mathbb{R}$ that rates bundles
- Define $h$ to choose the most appropriate bundle:

$$h(x) = \arg \max_{y \in Y(x-1)} f(x, y)$$
The Discriminant Function

- Impose additional structure on $f$:
  \[ f_w(x, y) = w_1 x_1 [y] + w_{-1}^T \psi(x_{-1}, y) \]

- $w = (w_1, w_{-1}) \in \mathbb{R}^{M+1}$ is a parameter vector to be learned
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The Payment Rule

- Ensure $w_1 > 0$ and let
  \[
  t_w(x, y) = -\left(\frac{w_{-1}}{w_1}\right)^T \psi(x_{-1}, y)
  \]

- agent-independent
- $h_w$ predicts the utility-maximizing bundle:
  \[
  h_w(x) = \arg\max_{y \in Y(x_{-1})} f_w(x, y) = \arg\max_{y \in Y(x_{-1})} w_1 x_1[y] + w_{-1}^T \psi(x_{-1}, y)
  \]
  \[
  = \arg\max_{y \in Y(x_{-1})} w_1 x_1[y] + w_{-1}^T \left(-\frac{w_1}{w_{-1}} t_w(x, y)\right)
  \]
  \[
  = \arg\max_{y \in Y(x_{-1})} (x_1[y] - t_w(x, y))
  \]

- Can ensure by translation that $w_{-1}^T \psi(x_{-1}, 0) = 0$, i.e., that payment for empty bundle is zero
Truthfulness and Regret

- Looks like the characterization of a strategyproof mechanism, but $h_w$ might not be feasible
- Also recall that we want to allocate according to $g$, not $h_w$
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**Lemma:** The ex-post regret for bidding truthfully in $(g, t_w)$ is

$$\frac{1}{w_1} \left( \max_{y' \in Y(x_{-1})} f_w(x, y') - f_w(x, g(x)) \right).$$

**Theorem:** If $h_w$ is exact, then $(g, t_w)$ is strategyproof.
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**Theorem:** If $h_w$ is exact, then $(g, t_w)$ is strategyproof.

- But: $h_w$ will not always be exact, we know it cannot be if $g$ is not monotonic
Regret and Generalization Error

- Generalization error of a classifier $h_w \in \mathcal{H}_\psi$:

$$R_P(h_w) = \int_{X \times Y} \Delta_x(y, h_w(x)) \, dP(x, y)$$

where $\Delta_x(y, y') = \frac{1}{w_1} (f_w(x, y') - f_w(x, y))$

**Theorem:** If $h_w$ minimizes generalization error then $t_w$ minimizes expected ex-post regret for truthful bidding.

- Amount a random agent can gain by lying when all others tell the truth, for valuations drawn from $P(X)$

- Different from (approximate) ex-ante and ex-interim equilibrium, rather provides an upper bound on the expected ex-interim gain
Support Vector Machines?

- Learn a discriminant function that maximizes the margin
- Binary setting: minimize generalization error in the limit
- Version with structured/multi-class output due to Joachims et al.
- Training by solving a quadratic optimization problem with linear constraints, can be done efficiently under certain conditions
- Training requires computation of inner products in the (high- or infinite-dimensional) feature space $\mathbb{R}^M$
- Kernel trick: choose $\psi$ carefully to ensure they can be computed efficiently from vectors in the original space
- Linear classification in $\mathbb{R}^M$ without any explicit calculations in $\mathbb{R}^M$
Summary

- Design of payment rules using margin-based classifier, given oracle access to valuation distribution and allocation rule.
- Exact classifier yields strategyproof payment rule, minimization of error implies minimization of expected ex-post regret.
- Experiments for 5 items, 2 to 6 agents, 200 training examples.

\[
\psi(x_{-1}, y) = \phi([x_2 \setminus y, \ldots, x_n \setminus y])
\]

- \(\phi\) corresponding to RBF kernel \(K(z, z') = \exp(-||z - z'||/2\sigma^2)\)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Accuracy</th>
<th>Average Regret</th>
<th>IR Violation</th>
</tr>
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<tbody>
<tr>
<td>Single item</td>
<td>96%</td>
<td>0.2%</td>
<td>2%</td>
</tr>
<tr>
<td>Single-minded</td>
<td>90%</td>
<td>1%</td>
<td>6%</td>
</tr>
<tr>
<td>Multi-minded, complements</td>
<td>94%</td>
<td>0.1%</td>
<td>3%</td>
</tr>
<tr>
<td>Multi-minded, substitutes</td>
<td>75%</td>
<td>2%</td>
<td>15%</td>
</tr>
</tbody>
</table>
Open Problems

- Possibly $-\mathbf{w}_1^T \psi(x_1, y) \geq x_1[y]$, failure of individual rationality
  - tradeoff between individual rationality and strategyproofness
  - both at the same time (only?) by deviation from $g$, e.g., by discarding $y$ and allocating $\emptyset$

- Training problem has $\Omega(|Y(x_1)|)$ constraints, exponential in $m$ in general
  - only polynomially many constraints matter, a separation oracle would suffice
  - when valuations can be represented succinctly, payment monotonicity would also suffice
  - more highly structured payment rules for restricted valuations

- More clever feature maps, e.g., to allow for generalization across different numbers of agents
Thank you!