TIME SERIES
Solutions to Exercise Sheet 6

1. (a) Seasonal ARMA$(1,1)_{12}$.
   (b) Seasonal MA$(2)_4$.
   (c) Non-seasonal AR(1).
   (d) Seasonal ARIMA$(2,0,1)\times(0,1,0)_{12}$.
   (e) Non-seasonal ARIMA$(2,1,0)$.

2. A seasonal AR$(1)_4$ model can be written as $(1-\Phi B^4)X_t = Z_t$ or $\Phi(B^4)X_t = Z_t$. To obtain the autocorrelation function, it is convenient to represent it as a linear process of the form

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \psi(B) Z_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Then we can write

$$X_t = \psi(B) Z_t = \psi(B)\Phi(B^4) X_t,$$

which means that

$$1 = \psi(B)\Phi(B^4),$$

or, in terms of polynomials in B,

$$1 = (\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \psi_4 B^4 + \psi_5 B^5 + \ldots)(1-\Phi B^4).$$

Expanding yields

$$\begin{aligned} 1 &= \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + (\psi_4 - \Phi)B^4 + (\psi_5 - \psi_1\Phi)B^5 + (\psi_6 - \psi_2\Phi)B^6 \\ &\quad +(\psi_7 - \psi_3\Phi)B^7 + (\psi_8 - \psi_4\Phi)B^8 + \ldots. \end{aligned}$$

Equating coefficients of B^j on the LHS and RHS of this equation, we obtain

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= \psi_2 = \psi_3 = 0 \\ \psi_4 &= \Phi \\ \psi_5 &= \psi_1\Phi = 0 \\ \psi_6 &= \psi_2\Phi = 0 \\ \psi_7 &= \psi_3\Phi = 0 \\ \psi_8 &= \psi_4\Phi = \Phi^2 \\ &\vdots \end{aligned}$$

In other words, we see that

$$\psi_j = \begin{cases} 0 & \text{if } j \neq 4k, k = 1,2,\ldots, \\ \Phi^k & \text{if } j = 4k, k = 0,1,2,\ldots. \end{cases}$$

Since X_t is a zero-mean process, by Corollary 3.1 in Section 3.4 of the lecture notes, we have

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+\tau},$$

which can be written as

$$\gamma(\tau) = \begin{cases} \sigma^2 \sum_{j=0}^{\infty} \psi_{4j}\psi_{4j+\tau} = \sigma^2 \sum_{j=0}^{\infty} \Phi^j \Phi^{j+\tau} & \text{if } \tau = 4k, k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $\tau = 4k, k = 0, 1, 2, \ldots$, we obtain

$$\gamma(\tau) = \sigma^2 \Phi^\tau \sum_{j=0}^{\infty} \Phi^{2j} = \frac{\sigma^2 \Phi^\tau}{1 - \Phi^2}.$$

By dividing $\gamma(\tau)$ by $\gamma(0)$, we obtain the required result.

3. A seasonal ARIMA$(1,1,1) \times (0,1,1)_{12}$ model can be written as

$$(1 - \phi B)(1 - B^{12})(1 - B)X_t = \alpha + (1 + \Theta B^{12})(1 + \theta B)Z_t.$$

When expanded, we have the form

$$\{1 - (\phi + 1)B + \phi B^2 - B^{12} + (\phi + 1)B^{13} - \phi B^{14}\}X_t = \alpha + (1 + \theta B + \Theta B^{12} + \Theta\theta B^{13})Z_t,$$

or

$$\begin{aligned} X_t &= \alpha + (\phi + 1)X_{t-1} - \phi X_{t-2} + X_{t-12} - (\phi + 1)X_{t-13} + \phi X_{t-14} + Z_t + \theta Z_{t-1} \\ &\quad + \Theta Z_{t-12} + \Theta\theta Z_{t-13}. \end{aligned}$$