Appendix

A.1. Vector identities

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}, \qquad \text{(A.1)}$$

$$\nabla \wedge \nabla \phi = 0, \qquad \nabla \cdot (\nabla \wedge \boldsymbol{F}) = 0, \qquad \text{(A.2, A.3)}$$

$$\nabla \cdot (\phi \boldsymbol{F}) = \phi \nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla \phi, \qquad \text{(A.4)}$$

$$\nabla \wedge (\phi \boldsymbol{F}) = \phi \nabla \wedge \boldsymbol{F} + (\nabla \phi) \wedge \boldsymbol{F}, \qquad \text{(A.5)}$$

$$\nabla \wedge (\boldsymbol{F} \wedge \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}(\nabla \cdot \boldsymbol{F}), \qquad \text{(A.6)}$$

$$\nabla \cdot (\boldsymbol{F} \wedge \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \wedge \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \wedge \boldsymbol{G}), \qquad \text{(A.7)}$$

$$\nabla(\boldsymbol{F} \cdot \boldsymbol{G}) = \boldsymbol{F} \wedge (\nabla \wedge \boldsymbol{G}) + \boldsymbol{G} \wedge (\nabla \wedge \boldsymbol{F}) + (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F}, \qquad \text{(A.8)}$$

$$(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} = (\nabla \wedge \boldsymbol{F}) \wedge \boldsymbol{F} + \nabla(\tfrac{1}{2}F^2), \qquad \text{(A.9)}$$

$$\nabla^2 \boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla \wedge (\nabla \wedge \boldsymbol{F}). \qquad \text{(A.10)}$$

A.2. Two properties of the gradient operator ∇

Let $\phi(\boldsymbol{x})$ be some scalar function of $\boldsymbol{x}$, and let $\mathrm{d}\phi/\mathrm{d}s$ be its rate of change, with distance s, in the direction of some unit vector $\boldsymbol{t}$. Then

$$\mathrm{d}\phi/\mathrm{d}s = \boldsymbol{t} \cdot \nabla \phi. \qquad \text{(A.11)}$$

For this very reason, the line integral of $\nabla\phi$ along some curve C is equal to the difference in ϕ between the two end-points of the curve:

$$\int_C \nabla \phi \cdot \mathrm{d}\boldsymbol{x} = [\phi]_C. \qquad \text{(A.12)}$$

A.3. The divergence theorem

Let the region V be bounded by a simple closed surface S with unit outward normal $\boldsymbol{n}$. Then

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \nabla \cdot \boldsymbol{F} \, \mathrm{d}V. \qquad \text{(A.13)}$$

In suffix notation, and using the summation convention, this takes the form

$$\int_S F_j n_j \, \mathrm{d}S = \int_V \frac{\partial F_j}{\partial x_j} \, \mathrm{d}V.$$

There are many identities which may be derived from the divergence theorem. The identity

$$\int_S \phi \boldsymbol{n} \, \mathrm{d}S = \int_V \nabla \phi \, \mathrm{d}V \qquad \text{(A.14)}$$

is particularly valuable, and may be written

$$\int_S \phi n_j \, \mathrm{d}S = \int_V \frac{\partial \phi}{\partial x_j} \, \mathrm{d}V. \qquad \text{(A.15)}$$

The following are immediate consequences:

$$\int_S F_i n_j \, \mathrm{d}S = \int_V \frac{\partial F_i}{\partial x_j} \, \mathrm{d}V, \qquad \int_S T_{ij} n_j \, \mathrm{d}S = \int_V \frac{\partial T_{ij}}{\partial x_j} \, \mathrm{d}V, \qquad \text{(A.16, A.17)}$$

$$\int_S u_i v_j n_j \, \mathrm{d}S = \int_V \frac{\partial}{\partial x_j}(u_i v_j) \, \mathrm{d}V. \qquad \text{(A.18)}$$

Other identities derivable from the divergence theorem include:

$$\int_S \boldsymbol{F} \wedge \boldsymbol{n} \, \mathrm{d}S = -\int_V \nabla \wedge \boldsymbol{F} \, \mathrm{d}V, \qquad \int_S \boldsymbol{n} \cdot \nabla \phi \, \mathrm{d}S = \int_V \nabla^2 \phi \, \mathrm{d}V, \qquad \text{(A.19, A.20)}$$

$$\int_S \phi \frac{\partial \psi}{\partial n} \, \mathrm{d}S = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, \mathrm{d}V, \qquad \text{(A.21)}$$

$$\int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \mathrm{d}S = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, \mathrm{d}V. \qquad \text{(A.22)}$$

A.4. Stokes's theorem

Let C be a simple closed curve spanned by a surface S with unit normal $\boldsymbol{n}$. Then

$$\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla\wedge \boldsymbol{F})\cdot \boldsymbol{n}\,\mathrm{d}S, \tag{A.23}$$

where the line integral is taken in an appropriate sense, according to that of $\boldsymbol{n}$ (see Fig. A.1).

Green's theorem in the plane may be viewed as a special case of Stokes's therorem, with $\boldsymbol{F}=[u(x,y), v(x,y), 0]$. If C is a simple closed curve in the x–y plane, and S denotes the region enclosed by C, then

$$\int_C u\,\mathrm{d}x + v\,\mathrm{d}y = \int_S \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y. \tag{A.24}$$

A useful identity derivable from Stokes's theorem is

$$\int_C \phi\,\mathrm{d}\boldsymbol{x} = -\int_S (\nabla\phi)\wedge \boldsymbol{n}\,\mathrm{d}S. \tag{A.25}$$

A.5. Orthogonal curvilinear coordinates

Let u, v, and w denote a set of orthogonal curvilinear coordinates, and let $\boldsymbol{e}_u$, $\boldsymbol{e}_v$ and $\boldsymbol{e}_w$ denote unit vectors parallel to the coordinate lines and in the directions of increase of u, v, and w respectively. Then

$$\boldsymbol{e}_u=\boldsymbol{e}_v\wedge\boldsymbol{e}_w, \qquad \text{etc.,}$$

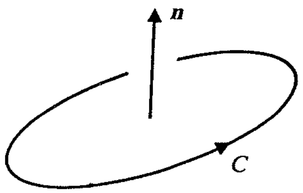

Fig. A.1.

and

$$\delta\boldsymbol{x}=h_1\,\delta u\boldsymbol{e}_u+h_2\,\delta v\boldsymbol{e}_v+h_3\,\delta w\boldsymbol{e}_w,$$

where

$$h_1=|\partial\boldsymbol{x}/\partial u|, \qquad \text{etc.}$$

Furthermore,

$$\nabla\phi=\frac{1}{h_1}\frac{\partial\phi}{\partial u}\boldsymbol{e}_u+\frac{1}{h_2}\frac{\partial\phi}{\partial v}\boldsymbol{e}_v+\frac{1}{h_3}\frac{\partial\phi}{\partial w}\boldsymbol{e}_w, \tag{A.26}$$

$$\nabla\cdot\boldsymbol{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3F_u)+\frac{\partial}{\partial v}(h_3h_1F_v)+\frac{\partial}{\partial w}(h_1h_2F_w)\right], \tag{A.27}$$

$$\nabla\wedge\boldsymbol{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_u & h_2\boldsymbol{e}_v & h_3\boldsymbol{e}_w\\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w}\\ h_1F_u & h_2F_v & h_3F_w \end{vmatrix}. \tag{A.28}$$

For *cylindrical polar coordinates* (Fig. A.2)

$$u=r, \qquad v=\theta, \qquad w=z,$$
$$h_1=1. \qquad h_2=r, \qquad h_3=1.$$

For *spherical polar coordinates* (Fig. A.3)

$$u=r, \qquad v=\theta, \qquad w=\phi,$$
$$h_1=1, \qquad h_2=r, \qquad h_3=r\sin\theta.$$

A.6. Cylindrical polar coordinates

Cylindrical polar coordinates (r, θ, z) are such that

$$x_1=r\cos\theta, \qquad x_2=r\sin\theta, \qquad x_3=z,$$

as in Fig. A.2. Clearly,

$$\delta\boldsymbol{x}=\delta r\boldsymbol{e}_r+r\,\delta\theta\boldsymbol{e}_\theta+\delta z\boldsymbol{e}_z$$

and

$$\boldsymbol{e}_r=\cos\theta\,\boldsymbol{e}_1+\sin\theta\,\boldsymbol{e}_2, \qquad \boldsymbol{e}_\theta=-\sin\theta\,\boldsymbol{e}_1+\cos\theta\,\boldsymbol{e}_2, \qquad \boldsymbol{e}_z=\boldsymbol{e}_3.$$

The unit vectors do not change with r or z, but

$$\frac{\partial\boldsymbol{e}_r}{\partial\theta}=\boldsymbol{e}_\theta, \qquad \frac{\partial\boldsymbol{e}_\theta}{\partial\theta}=-\boldsymbol{e}_r, \qquad \frac{\partial\boldsymbol{e}_z}{\partial\theta}=0. \tag{A.29}$$

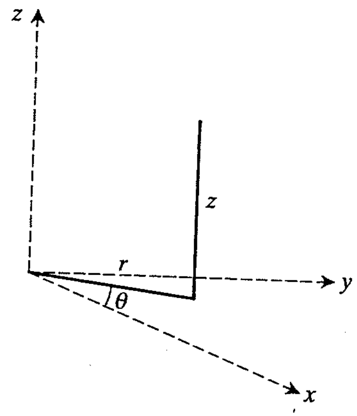

Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z,$$ (A.3

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z},$$ (A.3

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix},$$ (A.3

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},$$ (A.3

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}.$$ (A.3

The Navier–Stokes equations in cylindrical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right),$$

(A.35)

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0.$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{A.36}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

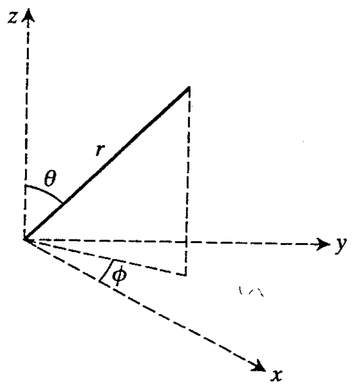

Fig. A.3. Spherical polar coordinates.

A.7. Spherical polar coordinates

Spherical polar coordinates (r, θ, ϕ) are such that

$$x_1 = r \sin\theta \cos\phi, \qquad x_2 = r\sin\theta\sin\phi, \qquad x_3 = r\cos\theta,$$

as in Fig. A.3. Clearly,

$$\delta\boldsymbol{x} = \delta r \boldsymbol{e}_r + r\delta\theta\boldsymbol{e}_\theta + r\sin\theta\,\delta\phi\boldsymbol{e}_\phi$$

and

$$\begin{aligned}\boldsymbol{e}_r &= \sin\theta\cos\phi\,\boldsymbol{e}_1 + \sin\theta\sin\phi\,\boldsymbol{e}_2 + \cos\theta\,\boldsymbol{e}_3,\\ \boldsymbol{e}_\theta &= \cos\theta\cos\phi\,\boldsymbol{e}_1 + \cos\theta\sin\phi\,\boldsymbol{e}_2 - \sin\theta\,\boldsymbol{e}_3,\\ \boldsymbol{e}_\phi &= -\sin\phi\,\boldsymbol{e}_1 + \cos\phi\,\boldsymbol{e}_2.\end{aligned}$$

The unit vectors do not change with r, but

$$\begin{gathered}\partial\boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta, \qquad \partial\boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r, \qquad \partial\boldsymbol{e}_\phi/\partial\theta = 0,\\ \partial\boldsymbol{e}_r/\partial\phi = \sin\theta\,\boldsymbol{e}_\phi, \qquad \partial\boldsymbol{e}_\theta/\partial\phi = \cos\theta\,\boldsymbol{e}_\phi, \qquad \text{(A.37)}\\ \partial\boldsymbol{e}_\phi/\partial\phi = -\sin\theta\,\boldsymbol{e}_r - \cos\theta\,\boldsymbol{e}_\theta.\end{gathered}$$

Also,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\boldsymbol{e}_\phi, \qquad \text{(A.38)}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}, \qquad \text{(A.39)}$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\phi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}, \qquad \text{(A.40)}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \qquad \text{(A.41)}$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}. \qquad \text{(A.42)}$$

The Navier–Stokes equations in spherical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_ru_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}$$

$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right],$$

$$\frac{\partial u_\phi}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\phi + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi\cot\theta}{r}$$

$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left[\nabla^2u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0. \qquad \text{(A.43)}$$

The components of the rate-of-strain tensor are given by:

$$\begin{gathered}e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r},\\ e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},\\ 2e_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}, \qquad \text{(A.44)}\\ 2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),\\ 2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.\end{gathered}$$