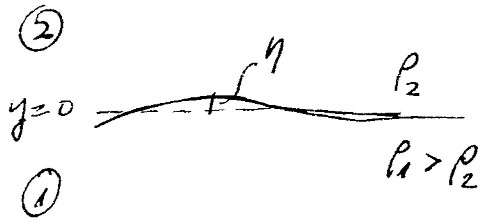
$\boxed{3.2}$



at the interface, where $p_2 = p_1 = p$,

$$-\rho_2\,\partial_t\phi_2 - \rho_2 g\eta = -\rho_1\,\partial_t\phi_1 - \rho_1 g\eta,$$

since the two regions share the common interface η.

η undulates about $y=0$. Therefore,

$$\rho_2\,\partial_t\phi_2 + \rho_2 g\eta = \rho_1\,\partial_t\phi_1 + \rho_1 g\eta\,, \quad y=0.$$

$\partial_x^2\phi + \partial_y^2\phi = 0$ still applies in both regions, giving for $\phi(x,y,t) = f(y)\sin(kx-\omega t)$,

$f_2(y) = De^{-ky}$ and $f_1(y) = Ce^{ky}$, k is assumed to be positive w/o loss of generality.

$$\therefore\ \phi_2 = De^{-ky}\sin(kx-\omega t)\,, \quad y>0$$
$$\phi_1 = Ce^{ky}\sin(kx-\omega t)\,, \quad y<0$$

Recall also that $\eta = A\cos(kx-\omega t)$

$$\Rightarrow -\rho_2(\omega D\cancel{e^{-ky}}^{\,1} - gA) = -\rho_1(\omega C\cancel{e^{ky}}^{\,1} - gA)\,, \quad y=0$$

$$\Rightarrow\ -\rho_2\omega D + \rho_1\omega C = \rho_1 gA - \rho_2 gA$$

Using $\partial_y\phi = \partial_t\eta$, $y=0$, get

$$\omega A = -kD = kC \Rightarrow \boxed{D=-C} \Rightarrow \boxed{A = \frac{k}{\omega}C = -\frac{k}{\omega}D}$$

$$\therefore\ \cancel{k}\,\omega(\rho_1+\rho_2) = (\rho_1-\rho_2)\,g\,\frac{k}{\omega}$$

$$\Rightarrow\ \omega^2 = gk\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right) \Rightarrow c^2 = \frac{g}{|k|}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)\ ; \quad c \equiv \frac{\omega}{k}$$

$\boxtimes$.

3.4

For instability, we need $\omega_N^2 < 0$ so that ω_N is imaginary. This leads to $\eta(x,t)$ to possess a component that grows exponentially in time:

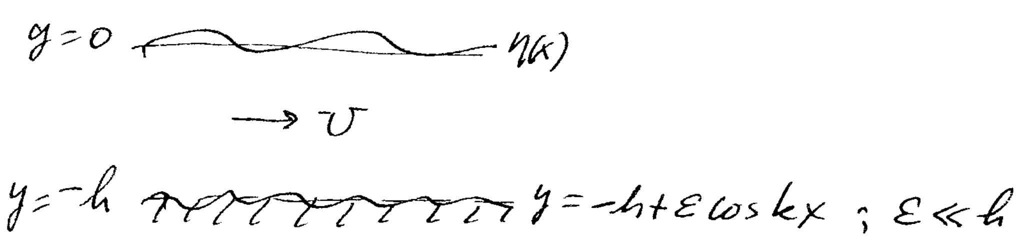
$$\omega_N^2 = \frac{N\pi}{a(\rho_1+\rho_2)}\left[-(\rho_2-\rho_1)g + T\frac{N^2\pi^2}{a^2}\right] ; \quad N = 1, 2, 3, \ldots$$

$$\Rightarrow \quad -(\rho_2-\rho_1)g + T\frac{N^2\pi^2}{a^2} < 0, \text{ since } \frac{N\pi}{a(\rho_1+\rho_2)} > 0.$$

$$\therefore \quad \underline{\underline{(\rho_2-\rho_1)g > \frac{N^2\pi^2T^2}{a^2} > \frac{\pi^2T^2}{a^2}}} \text{ for instability}$$

$\boxed{\times}$.

$\boxed{3.5}$



$$u = U + \partial_x \phi , \quad v = \partial_y \phi$$

$$u^2 = (U + \partial_x \phi)^2 \approx U^2 + 2U\partial_x \phi + O(\eta^2)$$

$\cancel{\partial_t \phi}^{\,0} + U\partial_x \phi + g\eta = 0$, $y = 0$, since U contribution can be subsumed into ϕ and noting that the flow is steady.

From $\partial_t \eta + u\partial_x \eta = v$, $y = \eta$, get

$U\partial_x \eta = U\dfrac{d\eta}{dx} = \partial_y \phi$, $y = 0$, again keeping only the 1st-O terms and noting steady flow.

At $y = -h$, we have $v = \partial_y \phi = U\partial_x y = -Uk\varepsilon \sin kx$;

that is, $\partial_y \phi = -Uk\varepsilon \sin kx$, $y = -h$.

In the interior, $\nabla^2 \phi = 0$ still applies, giving

$$\phi(x,y) = f(y)\sin(kx) \Rightarrow f(y) = Ce^{ky} + De^{-ky}.$$

Now, $\partial_y \phi = k(Ce^{ky} - De^{-ky})\sin(kx) = -Uk\varepsilon \sin(kx)$, $y = -h$

$$\Rightarrow -Ce^{-kh} + De^{kh} = U\varepsilon$$

$$U\frac{d\eta}{dx} = \partial_y \phi \Rightarrow \eta = \frac{1}{U}\int(\partial_y \phi)\,dx , \quad y = 0$$

$$= \frac{-k}{U}\cdot\frac{1}{k}(C - D)\cos kx = \left(\frac{D - C}{U}\right)\cos(kx), \quad y = 0$$

$$= -\frac{U}{g}\,\partial_x \phi\Big|_{y=0} = -\frac{U}{g}(C + D)\,k\cos(kx)$$

$$\Rightarrow \frac{U^2 k}{g}(C + D) = C - D \Rightarrow (\xi - 1)\,C = -(\xi + 1)\,D \; ; \; \xi \equiv \frac{U^2 k}{g}$$

⟶ next page

$$C = -\left(\frac{\xi+1}{\xi-1}\right) D \Rightarrow \left[\left(\frac{\xi+1}{\xi-1}\right) e^{-kh} + e^{kh}\right] D = U\varepsilon$$

$$\Rightarrow \boxed{D = \frac{U\varepsilon}{\left(\frac{\xi+1}{\xi-1}\right) e^{-kh} + e^{kh}}} \Rightarrow \boxed{C = -\frac{U\varepsilon(\xi+1)}{(\xi+1)e^{-kh} + (\xi-1)e^{kh}}}$$

Now, $\eta(x) = \frac{1}{U}(D-C)\cos(kx)$

$$= \frac{1}{\cancel{U}} \cdot \cancel{U}\varepsilon \left[\frac{(\xi-1) + (\xi-1)}{(\xi+1)e^{-kh} + (\xi-1)e^{kh}}\right] \cos(kx)$$

$$= \varepsilon \left[\frac{2\xi}{2\xi \cosh(kh) - 2\sinh(kh)}\right] \cos(kx)$$

$$\therefore \boxed{\eta(x) = \varepsilon \left[\frac{1}{\cosh(kh) - \frac{g}{U^2 k}\sinh(kh)}\right] \cos(kx)}$$

To obtain the "vericose" behaviour between the surface and the bed, we want

$$\cosh(kh) - \frac{g}{U^2 k}\sinh(kh) < 0$$

$$\Rightarrow U^2 < \frac{g}{k}\tanh(kh) \text{ , as required}$$

$\boxtimes$.