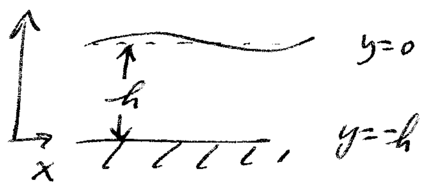
3.1

$\partial_x^2\phi + \partial_y^2\phi = 0 \quad ; \quad \phi = f(y)\sin(kx-\omega t)$



$$f'' - k^2 f = 0 \Rightarrow f = Ce^{ky} + De^{-ky}$$

· We may take $k>0$ without loss of generality. Now, unlike the situation in Sec 3.2, we must take into account the finite depth $-h$ of the fluid:

$$v(-h) = \partial_y\phi\big|_{-h} = k\left(Ce^{-kh} - De^{kh}\right) = 0 \Rightarrow \boxed{C = De^{2kh}}$$ (n.b., $C \neq C_p$, the phase vel – see below).

$$\Rightarrow \phi = \left(Ce^{ky} + Ce^{-2kh}\cdot e^{-ky}\right)\sin(kx-\omega t)$$

$$= Ce^{-kh}\left[e^{k(y+h)} + e^{-k(y+h)}\right]\sin(kx-\omega t).$$

· The B.C.'s for the top is the same as before:

$\partial_y\phi = \partial_t\eta \ , \quad y=0$

$\partial_t\phi + g\eta = 0 \ , \quad y=0.$

(Recall $\eta = A\cos(kx-\omega t)$.).

$$\partial_y\phi\big|_{y=0} = Ce^{-kh}\sin(kx-\omega t)\, k\left(e^{kh} - e^{-kh}\right) = +A\omega\sin(kx-\omega t)$$

$$\Rightarrow Cke^{-kh}\left(e^{kh} - e^{-kh}\right) = +A\omega$$

$$\eta\big|_{y=0} = -\frac{1}{g}\partial_t\phi\big|_{y=0} = +\frac{\omega}{g}\cos(kx-\omega t)\, Ce^{-kh}\left(e^{kh} + e^{-kh}\right) = A\cos(kx-\omega t)$$

$$\Rightarrow \frac{\omega}{g}Ce^{-kh}\left(e^{kh} + e^{-kh}\right) = A$$

$$\therefore \quad +\omega = \frac{kg}{\omega}\left(\frac{e^{kh} - e^{-kh}}{e^{kh} + e^{-kh}}\right) \Rightarrow \omega^2 = gk\tanh(kh)$$

$$\Rightarrow \frac{\omega^2}{k^2} = \boxed{C_p^2 = \frac{g}{k}\tanh(kh)} \quad \boxtimes.$$

· For validity $\frac{A}{L} \ll 1$, as before, where $L = \frac{2\pi}{k}$, but also $\frac{A}{h} \ll 1$, so that 'small-amplitude' linearisation can be made.

$$C = \frac{A\omega}{k}\, e^{kh} \frac{1}{e^{kh} - e^{-kh}} \Rightarrow \boxed{\phi = \frac{A\omega}{k}\left[\frac{e^{k(y+h)} + e^{-k(y+h)}}{e^{kh} - e^{-kh}}\right]\sin(kx-\omega t)}$$

$$u = \partial_x \phi = A\omega\left[\frac{e^{k(y+h)} + e^{-k(y+h)}}{e^{kh}-e^{-kh}}\right]\cos(kx-\omega t) \Rightarrow x' = -A\left[\frac{\cosh\{k(\bar{y}+h)\}}{\sinh(kh)}\right]\sin(k\bar{x}-\omega t)$$

$$v = \partial_y \phi = A\omega\underbrace{\left[\frac{e^{k(y+h)} - e^{-k(y+h)}}{e^{kh}-e^{-kh}}\right]}_{\text{note the change in sign in the numerator.}}\sin(kx-\omega t) \Rightarrow y' = +A\left[\frac{\sinh\{k(\bar{y}+h)\}}{\sinh(kh)}\right]\cos(k\bar{x}-\omega t).$$

$$\Rightarrow \left\{\frac{x'}{\cosh[k(\bar{y}+h)]}\right\}^2 + \left\{\frac{y'}{\sinh[k(\bar{y}+h)]}\right\}^2 = \left\{\frac{A}{\sinh(kh)}\right\}^2.$$

$\therefore$ particle paths are ellipses which flatten with depth: