TEST 1 SOLUTION

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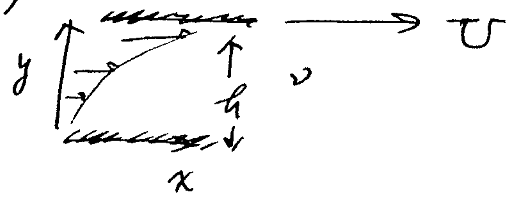
(a) NSE: $\partial_t \vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}\left(\frac{p}{\rho}\right) + \nu\nabla^2\vec{u}$; $\vec{\nabla}\cdot\vec{u} = 0$

v & w components of the eq'n gives $p = p_0 = \text{const}$.
In the u component direction, we have

$$\partial_t u + \cancel{u\partial_x u}^{0} + \cancel{v}^{0}\partial_y u + \cancel{w\partial_z u}^{0} = -\cancel{\partial_x\left(\tfrac{p}{\rho}\right)}^{0} + \nu\left(\cancel{\partial_x^2}^{0} + \partial_y^2 + \cancel{\partial_z^2}^{0}\right)u \;;\; u = u(y,t)$$

$\therefore$ $\partial_t u = \nu\partial_y^2 u$ ⊠.

(b)



$u(y,0) = 0$, $0 < y < h$
$u(0,t) = 0$, $t > 0$
$u(h,t) = U$, $t > 0$

(c) $u(y,t) = u_0(y) + u_1(y,t)$; $\cancel{\partial_t u_0}^{0} = \nu\partial_y^2 u_0 \Rightarrow \boxed{u_0(y) = Ay + B}$

B.C. : $u_0(0) = \boxed{0 = B}$, $u_0(h) = U = Ah \Rightarrow \boxed{A = \frac{U}{h}}$

$\therefore$ $\boxed{u_0(y) = \frac{U}{h}y}$

(d) Since $u(y,0) = u_0(y) + u_1(y,0)$; $u_0(y) = \frac{U}{h}y$

$0 = \frac{U}{h}y + u_1(y,0)$, $0 < y < h$ $\Rightarrow \boxed{u_1(y,0) = -\frac{U}{h}y}$, $0 < y < h$

(e) Now, we can try separation of variables: $u_1(y,t) = F(y)\cdot G(t)$;
$\partial_t u_1 = \nu\partial_y^2 u_1$. This gives $F\partial_t G = \nu G\partial_y^2 F \Rightarrow \frac{1}{\nu G}\partial_t G = \frac{1}{F}\partial_y^2 F = -\lambda^2$; $\lambda = \text{consts.}$

$\Rightarrow G(t) \sim e^{-\lambda^2\nu t}$, $F(y) = C\sin\lambda y + D\cos\lambda y$

From B.C., we have $u_1(0) = \boxed{0 = D}\cancel{\cos\lambda y}^{1}$, $u_1(h) = C\sin\lambda h = 0$

$\Rightarrow \lambda h = n\pi$; $n \in \mathbb{Z}^+ \Rightarrow \boxed{\lambda = \frac{n\pi}{h}}$. Using superposition,

$\therefore$ $\boxed{u_1(y,t) = \sum_{n=1}^{\infty}{}' A_n \sin\left(\frac{n\pi}{h}y\right)\cdot e^{-\frac{n^2\pi^2}{h^2}\nu t}}$; $n \in \mathbb{Z}^+$

⊠.