Queen Mary, University of London

MIDTERM EXAMINATION

MTH5106 Dynamics of Physical Systems

13 Nov 2009, 11.10 - 11.50 (40 minutes)

This paper contains 4 questions. You should attempt ALL questions. Answer each one clearly and legibly, crossing out any false starts.

Calculators are NOT permitted in this examination.

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1. Answer the following:

(a) Given two vectors, **a** and **b**, how can one test whether the two are parallel? Explain why the test works. [10 marks]

Ans: take the cross product of the two vectors and see if it is equal to zero. This test works because the magnitude of the cross product is proportional to the $\sin$ of the angle between the two vectors.

(b) Given the vector field, $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}$, where k is a positive constant and $\mathbf{r}$ is the position vector, what is the curl of the field? [10 marks]

Ans: the curl is zero, since the field is "central" (i.e., only contains a radial component). Alternatively, one could directly perform the curl operation – making the substitution, $\mathbf{r} = (x, y, z)$; the curl is zero, since k is a constant.

2. Answer the following:

(a) Evaluate the number, $e^{0.1}$, correct to three decimal places. [Hint: the Taylor expansion of a function $f(x)$ about $x = 0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!},$$

where $f^{(n)}(0)$ is the n-th derivative of $f(x)$ evaluated at $x = 0$.] [10 marks]

Ans: $e^{0.1} \approx 1 + 0.1 + \frac{1}{2}(0.1)^2 = 1.1 + 0.01/2 = 1.105$

(b) A contour ψ is the locus of points of the same (constant) value c in a scalar field so that $\psi(\mathbf{r}) \equiv \psi(x, y, z) = c$. Show that $\nabla\psi$ is perpendicular to the contour, whose line element is $\mathrm{d}\mathbf{r}$. [10 marks]

Ans:

$$\mathrm{d}\psi = \frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y + \frac{\partial\psi}{\partial z}\mathrm{d}z \;=\; \nabla\psi \cdot \mathrm{d}\mathbf{r}.$$

Now, since $\psi = c$ on a contour, $\mathrm{d}\psi = \nabla\psi \cdot \mathrm{d}\mathbf{r} = 0$. Therefore $\nabla\psi$ is perpendicular to $\mathrm{d}\mathbf{r}$.

3. Do the following.

(a) State Newton's first law. [5 marks]

Ans: an object in uniform motion tends to stay in uniform motion, unless acted upon by an external force; an object at rest tends to stay at rest, unless acted upon by an external force.

(b) Show that an object of mass m, dropped from height h, hits the ground with speed $\sqrt{2gh}$. [10 marks]

Ans: this can be shown two ways, "kinematically" or "dynamically". That is, use $v^2 = v_0^2 + 2a(y - y_0)$, where $v_0 = y = 0$, $y_0 = h$ and $a = -g$. Alternatively, one can use $\Delta(KE) + \Delta(PE) = 0 \Rightarrow \Delta(\frac{1}{2}mv^2) = \Delta(mgh)$, assuming $mgh = 0$ at the ground.

(c) Describe three ways of assessing whether a given force $\mathbf{F}(\mathbf{r})$ is conservative. [15 marks]

Ans:

1. See if the work done in a circuit in $\mathbf{F}$ vanishes: $\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.
2. See if there exists some scalar field (function) $U(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$.
3. See if $\nabla \times \mathbf{F} = 0$.

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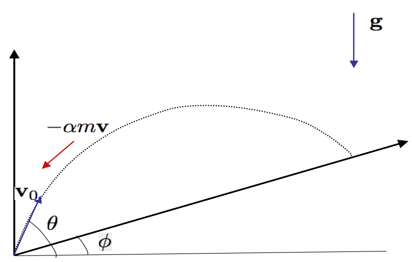

4. Consider the motion of a cannonball of mass m fired uphill. The hill slopes upward at an angle ϕ with respect to the horizontal. The cannon ball is fired upward with velocity, $\mathbf{v}_0 = (v_{0x}, v_{0y})$, at an angle θ with respect to the horizontal such that $\theta > \phi$. Both angles are less than 90°; gravity $\mathbf{g}$ points downward; and, there is air friction, which is directed opposite to the motion: $-\alpha m\mathbf{v} = -\alpha m(v_x, v_y)$, where α is a positive constant.

(a) Sketch the situation. [10 marks]

Ans: see the diagram above.

(b) What is the vertical component of the velocity when the cannonball is at the maximum height of its trajectory? [5 marks]

Ans: 0 (same as in *without* friction).

(c) What is the horizontal component of the cannonball's velocity $v_x(t)$, as a function of v_0 and θ, throughout its trajectory? [Hint: write down the differential equation for $v_x(t)$ and then solve it, subject to the initial condition.] [15 marks]

Ans: the equation for v_x is:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\alpha m v_x$$

$$\Rightarrow \quad \frac{\mathrm{d}v_x}{\mathrm{d}t} = -\alpha v_x \;\Rightarrow\; v_x(t) = v_{0x}e^{-\alpha t} = v_0\cos\theta\, e^{-\alpha t}.$$

(N.B., the form of $v_x(t)$ is not changed by the sloping ground – only the time the cannonball is in the air.)

[End of examination paper.]