MTH5106 – Dynamics of Physical Systems **Dr. J. Cho**

Problem Set 8 (4 questions)

Due Thursday, 17 Dec 2009, at (or before) 11:50 am in the Tutorial in ENG 325

N.B., your solutions may be handed in at the lecture just before the tutorial, if you are not planning to attend the above tutorial. THIS IS THE FINAL DPS COURSEWORK FOR THE TERM.

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1. Do the following:

   (i) Briefly, describe a "central force". Give a mathematical example.

   (ii) Using the equation of motion in polar coordinates, show that Kepler's second law holds for *any* central force.

2. Use the equation of motion, $m\,(\ddot{r}-r\dot{\theta}^2) = -\mu(r)$, and the constant of integration, $r^2\dot{\theta} = h$, to do the following:

   (i) Derive the differential equation for a planet's orbit around the Sun,
   $$\frac{d^2u}{d\theta^2} + u = \frac{\mu(\frac{1}{u})}{mh^2u^2},$$
   where $u = 1/r$. This equation can be interpreted as a nonlinearly-forced harmonic oscillator equation [e.g., $u'' + u = \mathcal{F}(u)$, where $u' \equiv \mathrm{d}u/\mathrm{d}\theta$].

   (ii) For the special central force, $\mu(r) = GmM/r^2$, where m is the mass of the planet, M is the mass of the Sun and G is the universal gravitational law constant, show that the above equation for the planet's orbit can be reduced (transformed) to a harmonic oscillator equation with a *constant* forcing, GM/h^2.

3. For the differential equation obtained in Problem 2(ii), do the following:

   (i) Write down the general solution in terms of trigonometric functions.

   (ii) Given the initial conditions, $u(0) = 1/d$ and $u'(0) = 0$ (recall $u(\theta) \equiv 1/r(\theta)$), show that the solution to the equation, subject to the initial condition, is an ellipse: $1/r = \psi\cos\theta + \phi$, where ψ and ϕ are constants – thus proving Kepler's first law. [Note, the standard form for r is: $r = a(1-e^2)/(1+e\cos\theta)$, where $e \equiv 1-(b/a)^2$ is the eccentricity and a and b are the semi-major and semi-minor axes, respectively.]

4. A simple model of the hydrogen atom is an electron of mass m orbiting a proton of mass M much like a planet orbiting the Sun. However, the electron and proton have charges $-q$ and $+q$ on them, respectively, and their motion is governed by a (slightly) different central force, the Coulomb force $-Kq^2/r^2$. Here K is a constant.

   (i) Use the force balance to determine the velocity of the electron going around the proton in a circular orbit with radius r. [Hint: what force balances the inward force on the electron?]

   (ii) What is the angular momentum L? That is, derive the expression for L in terms of the quantities already given: K, q, m and r.

   (iii) In a first attempt to incorporate quantum theory to the above classical picture, the angular momentum is assumed to take on only integer values of $\hbar$, the Planck's constant, in such a way that $L = n\hbar$, where $n \in \mathcal{Z}^+$ (i.e., a positive integer). Note the units of $\hbar$ is [J·s]. Obtain L in terms of $\hbar$, n, K, q and m.