MTH5106, Dynamics of Physical Systems **Dr. J. Cho**
Solutions to Problem Set 8

1. (i) A central force $\mathbf{F}$ is a force which is directed in the radial direction (only) with a component (magnitude) which is function only of the radial distance r. An example is $\mathbf{F} = f(r)\hat{\mathbf{r}}$. Another is $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}}$, where G, M and m are constants.

   (ii) For any central force,

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 = 2\dot{r}\dot{\theta} + r\ddot{\theta} \;\Rightarrow\; 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \;\Rightarrow\; r^2\dot{\theta} = \text{const} = h.$$

   But, since the change in infinitesimal area δA per unit time δt as $\delta t \to 0$ (so that $\delta r \to 0$ and $\delta\theta \to 0$) is:

$$\frac{1}{2}r^2\frac{\delta\theta}{\delta t} \approx \frac{\delta A}{\delta t} \;\Rightarrow\; \dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2},$$

   the area swept out per unit time is a constant. Hence, "the equal area rule" (Kepler's second law) is simply a consequence of the force being central.

2. We are given $m(\ddot{r} - r\dot{\theta}^2) = -f(r)$ and $r^2\dot{\theta} = h$.

   (i) Since $r = r(\theta(t))$,

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right).$$

   Now, letting $u = 1/r$, we get $\dot{r} = -h\,du/d\theta$. Also, $h/r = hu = r\dot{\theta} \Rightarrow r\dot{\theta}^2 = h^2u^3$. In addition,

$$\begin{aligned}
\ddot{r} &= \frac{d}{dt}\dot{r} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = -h^2u^2\frac{d^2u}{d\theta^2} \\
&\Rightarrow\; m(\ddot{r} - r\dot{\theta}^2) = -f(r) = m\ddot{r} - mr\dot{\theta}^2 \\
&\Rightarrow\; -mh^2u^2\frac{d^2u}{d\theta^2} - mh^2u^3 = -f\left(\frac{1}{u}\right) \\
&\Rightarrow\; \frac{d^2u}{d\theta^2} + u = \frac{f\left(\frac{1}{u}\right)}{mh^2u^2}.
\end{aligned}$$

   (ii) If $f(r) = GMm/r^2 = GMmu^2$, then the differential equation for the orbit obtained in part (a) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{GMmu^2}{mh^2u^2} = \frac{GM}{h^2},$$

   an equation for the harmonic oscillator with constant forcing.

3. (i) The general solution to the equation obtained in part (b) is:

$$u(\theta) = A\cos\theta + B\sin\theta + GM/h^2,$$

where A and B are to be determined from the initial condition.

(ii) Using the given initial condition, we have

$$\begin{aligned} \frac{1}{d} &= A + \frac{GM}{h^2} \Rightarrow A = \frac{1}{d} - \frac{GM}{h^2} \\ \frac{du}{d\theta} &= -A\sin\theta + B\cos\theta \Rightarrow B = 0 \\ &\Rightarrow u(\theta) = \left(\frac{1}{d} - \frac{GM}{h^2}\right)\cos\theta + \frac{GM}{h^2} = \frac{1}{r}. \end{aligned}$$

Therefore, $1/r = \psi\cos\theta + \phi$, with $\psi = (1/d) - \phi$ and $\phi = GM/h^2$.

4. The electron feels the given central force, which is balanced by the centrifugal force.

(i) Therefore,

$$\frac{Ke^2}{r^2} = \frac{mv^2}{r} \;\Rightarrow\; v^2 = \frac{Ke^2}{mr} \;\Rightarrow\; v = \left(\frac{Ke^2}{mr}\right)^{\frac{1}{2}}.$$

(ii) The angular momentum,

$$L = mvr = \frac{mK^{\frac{1}{2}}er}{m^{\frac{1}{2}}r^{\frac{1}{2}}} = (Ke^2mr)^{\frac{1}{2}}.$$

(iii) Given $L = n\hbar$, we have

$$L = n\hbar = (Ke^2mr)^{\frac{1}{2}} \;\Rightarrow\; n^2\hbar^2 = Ke^2mr \;\Rightarrow\; r(n) = \frac{n^2\hbar^2}{Ke^2m}.$$