MTH5106 – Dynamics of Physical Systems **Dr. J. Cho**

Problem Set 7 (4 questions) **Due Tuesday, 8 Dec 2009, at 12:00 in the Lec Room**

Your solutions should be handed in *before* start of the lecture on the due date. Given $x = x(t)$, note the following notational equivalences: $\mathrm{d}x/\mathrm{d}t = x' = \dot{x}$ and $\mathrm{d}^2x/\mathrm{d}t^2 = x'' = \ddot{x}$.

==

1. The following linear differential equation describes a forced, harmonic oscillator:

$$\ddot{x} + \omega^2 x = Ft,$$

where ω^2 and F are positive constants and with the initial condition $x(0) = 0$ and $\dot{x}(0) = 0$.

(i) What are the units of F?

(ii) What is the homogeneous solution to the given equation? What is the particular solution?

(iii) Now, what is the solution subject to the given initial condition?

(iv) Sketch the solution. Describe what is happening.

2. The following differential equation describes a periodically forced, harmonic oscillator:

$$\ddot{x} + \omega^2 x = G \sin \Omega t,$$

where ω^2, G and Ω are positive constants and with the initial condition $x(0) = 0$ and $\dot{x}(0) = 0$.

(i) What is the homogeneous solution to the given equation? What is the particular solution?

(ii) Now, what is the solution subject to the given initial condition?

(iii) Describe what happens when $\omega = \Omega$; what happens to x?

(iv) What happen at early times, when $\Omega t \ll 1$; how does x behave with t at early times? [Hint: go to third order in small Ωt.]

3. Given $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\theta} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$, where $\theta = \theta(t)$,

(i) Show that $\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\theta}\hat{\mathbf{r}}$.

(ii) Using the definition of the position vector, $\mathbf{r} = r\hat{\mathbf{r}}$, show that $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$, where $\mathbf{v}$ is the velocity.

(iii) Show that $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$, where $\mathbf{a}$ is the acceleration.

(iv) Using the acceleration given in part (iii), show that uniform circular motion with constant angular velocity (i.e., $\dot{\theta} =$ const. and $r =$ const.) leads to the centripetal acceleration, $\mathbf{a}_c = -(v_\theta^2/r)\hat{\mathbf{r}}$, where $v_\theta = r\dot{\theta}$ is the tangential speed.

[continued on the next page..]

4. A hanging pendulum consists of a mass m and a massless rigid rod of length l. There is gravity $\mathbf{g}$, which points straight down; but, the pendulum is *not* forced by any other external agencies.

   (i) Sketch the situation when the mass is displaced by an angle θ in the counter-clockwise direction. Be sure to indicate the forces involved.

   (ii) Using the results of Question 3, show that the governing equation of motion can be written as:

   $$\ddot{\theta} + \frac{g}{l} \sin \theta = 0.$$

   [N.B., l is a constant, since the rod is rigid.]

   (iii) Show that for small displacement angle, the equation in part (ii) can be written as:

   $$\ddot{\theta} + \omega^2 \theta = 0,$$

   where $\omega^2 = g/l$.

   (iv) Write down the general solution to the equation in part (iii).