MTH5106-DP&S', CW7 - SoL's

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1 $\ddot{x} + \omega^2 x = Ft$; $\omega^2, F > 0$, $x(0) = 0$ and $\dot{x}(0) = 0$

(i) Compare "Ft" term with "$\ddot{x}$" term:

$$[F]\cdot s = \frac{m}{s^2} \Rightarrow \boxed{[F] = m/s^3}$$

(ii) $\boxed{x_h = A\sin\omega t + B\cos\omega t}$

$x_p = ct$; $c = \text{const.}$ $\Rightarrow \omega^2 c \cancel{t} = F\cancel{t} \Rightarrow c = \frac{F}{\omega^2}$

$$\Rightarrow \boxed{x_p = \left(\frac{F}{\omega^2}\right)t}$$

(iii) $x = x_h + x_p = A\sin\omega t + B\cos\omega t + \frac{F}{\omega^2}t$

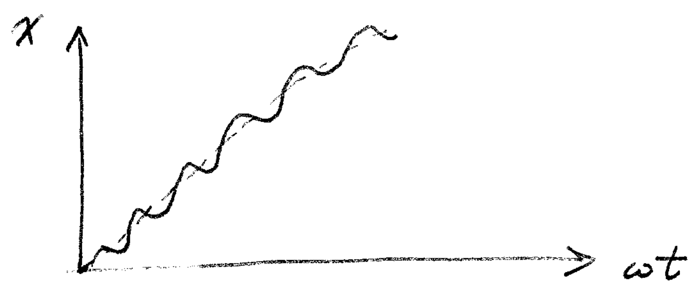
$x(0) = \boxed{0 = B}$

$\dot{x} = \omega A\cos\omega t - \omega B\sin\omega t + \frac{F}{\omega^2} \Rightarrow \dot{x}(0) = 0 = \omega A + \frac{F}{\omega^2}$

$$\Rightarrow \boxed{A = -\frac{F}{\omega^3}}$$

$$\Rightarrow x(t) = -\frac{F}{\omega^3}\sin\omega t + \frac{F}{\omega^2}t \Rightarrow \boxed{x(t) = \frac{F}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)}$$

(iv)



amplitude grows in time (sinusoidal oscillation superimposed on linear growth); frequency is ω.

$\boxed{2}$ $\ddot{x} + \omega^2 x = G \sin \Omega t$; $\omega^2, G, \Omega > 0$, $x(0) = 0$, $\dot{x}(0) = 0$

(i) $\boxed{x_h = A \sin \omega t + B \cos \omega t}$

$x_p = c \sin \Omega t \Rightarrow \dot{x} = c\Omega \cos \Omega t \Rightarrow \ddot{x} = -c\Omega^2 \sin \Omega t$

$\Rightarrow -c\Omega^2 \cancel{\sin \Omega t} + c\omega^2 \cancel{\sin \Omega t} = G \cancel{\sin \Omega t}$

$\Rightarrow c(\omega^2 - \Omega^2) = G \Rightarrow c = \dfrac{G}{\omega^2 - \Omega^2}$

$\Rightarrow \boxed{x_p = \dfrac{G}{\omega^2 - \Omega^2} \sin \Omega t}$

(ii) $x(t) = A \sin \omega t + B \cos \omega t + \dfrac{G}{\omega^2 - \Omega^2} \sin \Omega t$

$x(0) = \boxed{0 = B}$

$\dot{x}(t) = \omega A \cos \omega t - \omega B \sin \omega t + \dfrac{G\Omega}{\omega^2 - \Omega^2} \cos \Omega t$

$\dot{x}(0) = 0 = \omega A + \dfrac{G\Omega}{\omega^2 - \Omega^2} \Rightarrow \boxed{A = \dfrac{G(\Omega/\omega)}{\Omega^2 - \omega^2}}$

$\therefore \; x(t) = \dfrac{G(\Omega/\omega)}{\Omega^2 - \omega^2} \sin \omega t + \dfrac{G}{\omega^2 - \Omega^2} \sin \Omega t$

$$\boxed{x(t) = \left(\frac{G}{\omega^2 - \Omega^2}\right)\left[\sin \Omega t - \left(\frac{\Omega}{\omega}\right) \sin \omega t\right]}$$

(iii) When $\omega = \Omega$, $x \to \infty$.

(iv) Normally, when $\omega \neq \Omega$, we get $\boxed{x \approx \dfrac{G}{\omega^2 - \Omega^2}\left(\Omega t - \dfrac{\Omega}{\omega} \sin \omega t\right)}$ if $\Omega t \ll 1$ and $\omega t \ll 1$. However, if $\Omega \to \omega$, then $x \to$ "$G \cdot \dfrac{0}{0}$" and the problem becomes more subtle. If one goes to higher order,

$$x = \frac{G}{\omega^2 - \Omega^2}\left[\left(\Omega t - \frac{1}{6}\Omega^3 t^3\right) - \frac{\Omega}{\omega}\left(\omega t - \frac{1}{6}\omega^3 t^3\right)\right] + O((\Omega t)^5)$$

$= \dfrac{G}{\omega^2 - \Omega^2}\left[\dfrac{1}{6}\Omega t^3 (\omega^2 - \Omega^2)\right] = \boxed{\dfrac{G\Omega}{6} t^3}$ i.e., $x \sim t^3$ at early times.

$\boxed{3}$ $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$; $\theta = \theta(t)$

(i) $\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j} = \dot{\theta}\,\hat{\theta}$ $\boxtimes$

$\dot{\hat{\theta}} = -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j} = -\dot{\theta}\,\hat{r}$ $\boxtimes$

(ii) $\vec{r} = r\hat{r} \Rightarrow \vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\boxtimes$

(iii) $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$

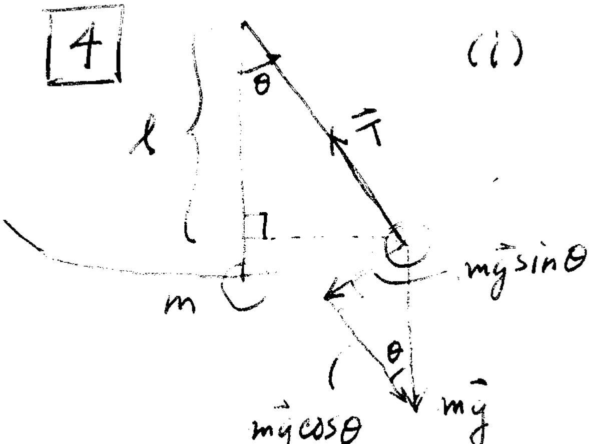
$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$ $\boxtimes$.

(iv) $r = \text{const}$ and $\dot{\theta} = \text{const} \Rightarrow \vec{a} = (\overset{0}{\cancel{\ddot{r}}} - r\dot{\theta}^2)\hat{r} + \underbrace{(2\overset{0}{\dot{r}}\dot{\theta} + r\overset{0}{\ddot{\theta}})}_{0}\hat{\theta}$

$\therefore\ \vec{a} = -r\dot{\theta}^2\hat{r} = -\dfrac{v_\theta^2}{r}\hat{r}$; $v_\theta = r\dot{\theta}$ $\boxtimes$.

$\boxed{4}$ (i)



(ii) From prob. 3, $m\vec{a} = m(\ddot{\ell} - \ell\dot{\theta}^2)\hat{r} + m(2\dot{\ell}\dot{\theta} + \ell\ddot{\theta})\hat{\theta}$; $r = \ell$.

In the $\hat{\theta}$: $m\ell\ddot{\theta} = -mg\sin\theta$, or $\ddot{\theta} + \left(\dfrac{g}{\ell}\right)\sin\theta = 0$ $\boxtimes$.

(iii) When θ is small, we get: $\ddot{\theta} + \omega^2\theta = 0$; $\omega^2 = \dfrac{g}{\ell}$ $\boxtimes$.

(iv) $\boxed{\theta(t) = A\sin\omega t + B\cos\omega t}$