MTH5106 – Dynamics of Physical Systems **Dr. J. Cho**

Problem Set 6 (4 questions) **Due Tuesday, 1 Dec 2009, at 12:00 in the Lec Room**

Your solutions should be handed in *before* start of the lecture on the due date. Given $x = x(t)$, note the following notational equivalences: $\mathrm{d}x/\mathrm{d}t = x' = \dot{x}$ and $\mathrm{d}^2x/\mathrm{d}t^2 = x'' = \ddot{x}$.

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1. Given the general first-order linear equation

$$\dot{x} + p(t)x = q(t),$$

with the initial condition $x(0) = x_0$,

(i) Show that the integrating factor, $I = e^{r(t)}$, where $r(t) = \int_0^t p(t')dt'$.

(ii) Obtain the solution, $x(t) = e^{-r(t)}[x_0 + \int_0^t q(t')e^{r(t')}dt']$.

(iii) Show that the solution in part (ii) indeed gives x_0 when $t = 0$.

(iv) Show that the solution in part (ii) is indeed the solution to the general equation by directly substituting the solution into the equation.

2. Given the forced, linear differential equation,

$$\dot{x} + 3x = 6t + 3,$$

(i) Sketch the *direction field* in (t, x)-space for arbitrary initial condition, $x(0) = x_0$; $x_0 \in \mathbb{R}$.

(ii) Obtain analytically the general solution for the arbitrary initial condition x_0. [Hint: see Question 1.]

(iii) Obtain the solution for the specific initial condition, $x_0 = -1$. Verify that your solution is correct by checking to see if it satisfy both the initial condition and the original differential equation.

(iv) What is the behaviour of the general solution at early times (i.e., at $t \to 0$)? Also, what is the behaviour of the general solution at late times (i.e., at $t \to \infty$)?

3. The following differential equation describes the motion of a damped, harmonic oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0,$$

where γ and ω^2 are positive constants and with the initial condition $x(0) = x_0$ and $\dot{x}(0) = 0$.

(i) What are the units of γ and ω?

(ii) What is the solution to the above differential equation, subject to the given initial condition, in the weakly damped case?

(iii) Sketch the solution.

(iv) Compare the solution with that in the undamped case. What happens to the amplitude when damping is introduced? What happens to the frequency?

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4. For the damped, harmonic oscillator equation,

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0,$$

without the initial condition specified:

(i) What is the general solution in the strongly damped case? (N.B., initial condition has not been specified; hence, the solution should contain two unspecified constants.)

(ii) Show that the solution describes two different decaying exponentials in the strongly damped case.

(iii) What is the general solution in the critically damped situation? Substitute your answer back into the original equation to show that it is indeed the solution.

(iv) Show how to obtain the general solution using the linear operator, $[\mathcal{D}^2 + 2\gamma\mathcal{D} + \omega^2]$, where $\mathcal{D} = d/dt$ and $\mathcal{D}^2 = d^2/dt^2$.