MTH5106-DPS CW6 SOL'NS

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1 $\dot{x} + p(t)\,x = q(t)$; $x = x(t)$, $x(0) = x_0$

(i) To show that $I = e^{\int_0^t p(t')dt'}$ is the integrating factor, let $r = \int_0^t p(t')dt'$, giving $I = e^r$, as suggested. Then,

$$e^r\dot{x} + xpe^r = qe^r \Rightarrow \frac{d}{dt}(xe^r) = e^r\dot{x} + xe^r\dot{r} \; ; \; r = r(t).$$

Now, $\frac{d}{dt}(xe^r) = qe^r$ if $\dot{r} = p$ by comparing with $e^r\dot{x} + xpe^r = qe^r$. Then, $\dot{r} = p \Rightarrow r(t) = \int_0^t p(t')dt'$.

Hence, $I = e^{\int_0^t p(t')dt'}$ ⊠.

(ii) From part (i), we have $\frac{d}{dt}(xe^r) = qe^r \Rightarrow xe^r\Big|_0^t = \int_0^t q(t')e^{r(t')}dt'$

$$\Rightarrow x(t)\,e^{r(t)} - x(0)\,\cancel{e^{r(0)}}^{\,1} = x(t)\,e^{r(t)} - x_0 = \int_0^t q(t')\,e^{r(t')}dt'$$

$$\Rightarrow \boxed{x(t) = e^{-r(t)}\left[x_0 + \int_0^t q(t')\,e^{r(t')}dt'\right]}$$

(iii) $x(t) = e^{-r(t)}\left[x_0 + \int_0^t q(t')\,e^{r(t')}dt'\right] \Rightarrow x(0) = 1\cdot\left[x_0 + 0\right] = x_0$

$\therefore\; x(0) = x_0$ ⊠.

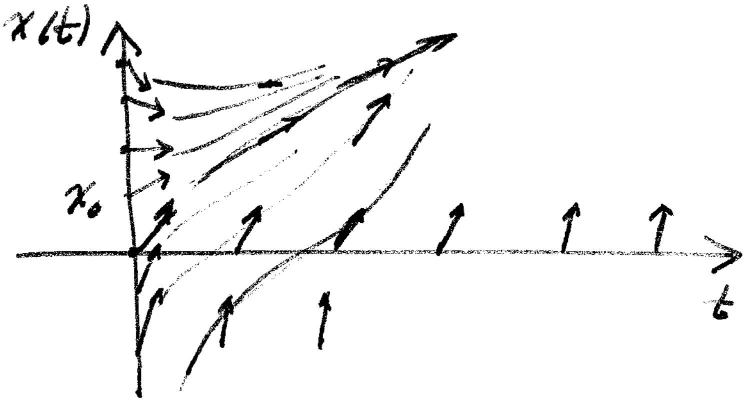
(iv) $x(t) = e^{-r(t)}\left[x_0 + \int_0^t q(t')\,e^{r(t')}dt'\right] \Rightarrow \dot{x} = e^{-r}(-\dot{r})[\quad] + e^{-r}qe^r$

$$\Rightarrow \dot{x} = -pe^{-r}\left[x_0 + \int_0^t q(t')\,e^{r(t')}dt'\right] + q$$

$\Rightarrow \underline{\dot{x} + px = q}$, since $x = e^{-r}[\quad]$ ⊠.

$\boxed{2}$ $\dot{x} + 3x = 6t + 3$

(i) $\frac{dx}{dt} = -3x + 6t + 3$



(ii) From prob. 1, $I = e^{\int_0^t 3\,dt'} = e^{3t}$ (i.e., $r = 3t$).

$$\Rightarrow x(t) = e^{-3t}\left[x_0 + \int_0^t (6t'+3)\,e^{3t'}\,dt'\right]$$

$$\therefore\ x(t) = e^{-3t}\left[x_0 + 6\int_0^t t' e^{3t'}\,dt' + 3\int_0^t e^{3t'}\,dt'\right]$$

$$= e^{-3t}\left[x_0 + \underbrace{2t\,e^{3t} - 2\int_0^t e^{3t'}\,dt'}_{\substack{\text{integration by parts:}\\ u=t \quad v=\frac{1}{3}e^{3t}\\ du=dt \quad dv=e^{3t}dt}} + 3\int_0^t e^{3t'}\,dt'\right]$$

$$\Rightarrow x(t) = x_0 e^{-3t} + 2t + \frac{1}{3}e^{-3t}\left(e^{3t'}\Big|_0^t\right)$$

$$= x_0 e^{-3t} + 2t + \frac{1}{3}\left(1 - e^{-3t}\right)$$

$$\therefore\ \boxed{x(t) = \frac{1}{3} + 2t + \left(x_0 - \frac{1}{3}\right)e^{-3t}}$$

(iii) $x_0 = -1 \Rightarrow \boxed{x(t) = \frac{1}{3} + 2t - \frac{4}{3}e^{-3t}}$

I.C. Check: $t=0 \Rightarrow x(0) = \frac{1}{3} - \frac{4}{3} = \boxed{-1}$ ✓

eq'n check:
$$\left.\begin{aligned} \dot{x} &= 2 + 4e^{-3t} \\ 3x &= 1 + 6t - 4e^{-3t} \end{aligned}\right\} \Rightarrow \boxed{\dot{x} + 3x = 6t + 3} \ ✓$$

(iv) $t \to 0 \Rightarrow x \approx \frac{1}{3} + 2t + \left(x_0 - \frac{1}{3}\right)(1-3t) = \frac{1}{3} + 2t + x_0 - 3x_0 t - \frac{1}{3} + t$

$\therefore\ \boxed{x \approx x_0 + 3(1-x_0)t}$ i.e., linear in t (as expected). $x_0 = 1$ is special (independent of t).

$t \to \infty \Rightarrow \boxed{x \approx 2t}$, as expected from sketch in part (i) - linear in t w/ slope 2.

$\boxed{3}$ $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$; γ, ω^2 positive constants and $x(0) = x_0$, $\dot{x}(0) = 0$.

(i) $[\gamma]\cdot\frac{m}{s} = \frac{m}{s^2} \Rightarrow \boxed{[\gamma] = \frac{1}{s}}$

$\omega^2 \cdot m = \frac{m}{s^2} \Rightarrow \boxed{[\omega] = \frac{1}{s}}$

(ii) obtain the general sol'n using the trial sol'n, $x \sim e^{\lambda t}$

$\Rightarrow \dot{x} \sim \lambda e^{\lambda t}$ and $\ddot{x} \sim \lambda^2 e^{\lambda t}$

$\Rightarrow$ the characteristic eq'n, $\boxed{\lambda^2 + 2\gamma\lambda + \omega^2 = 0}$

$\Rightarrow \lambda_\pm = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = \boxed{-\gamma \pm \sqrt{\gamma^2 - \omega^2}}$

$\therefore \boxed{x = Ae^{\lambda_+ t} + Be^{\lambda_- t}}$; "$\pm$" subscript corresponds to the $\pm$ branch of the λ roots.

$\Rightarrow \dot{x} = \lambda_+ A e^{\lambda_+ t} + \lambda_- B e^{\lambda_- t} \Rightarrow \dot{x}(0) = \lambda_+ A + \lambda_- B = 0$

also, $x(0) = A + B = x_0 \Rightarrow \boxed{B = x_0 - A} \Rightarrow \lambda_+ A + \lambda_-(x_0 - A) = 0$

$\Rightarrow (\lambda_+ - \lambda_-)A = -\lambda_- x_0 \Rightarrow \boxed{A = \left(\frac{\lambda_-}{\lambda_- - \lambda_+}\right)x_0} \Rightarrow B = \left(1 - \frac{\lambda_-}{\lambda_- - \lambda_+}\right)x_0$

$\Rightarrow \boxed{B = \left(\frac{-\lambda_+}{\lambda_- - \lambda_+}\right)x_0}$ $\therefore$ $\boxed{x(t) = \left(\frac{x_0}{\lambda_- - \lambda_+}\right)\left(\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}\right)}$

In the weakly damped case, $\lambda_\pm \in \mathbb{C}$, since $\gamma^2 < \omega^2$.

$$\therefore x(t) = -\frac{x_0}{2i\sqrt{\omega^2-\gamma^2}}\left[-(\gamma + i\sqrt{\omega^2-\gamma^2})e^{-\gamma t}\cdot e^{i\sqrt{\omega^2-\gamma^2}\,t} - (-\gamma + i\sqrt{\omega^2-\gamma^2}) \times e^{-\gamma t}\cdot e^{-i\sqrt{\omega^2-\gamma^2}\,t}\right]$$

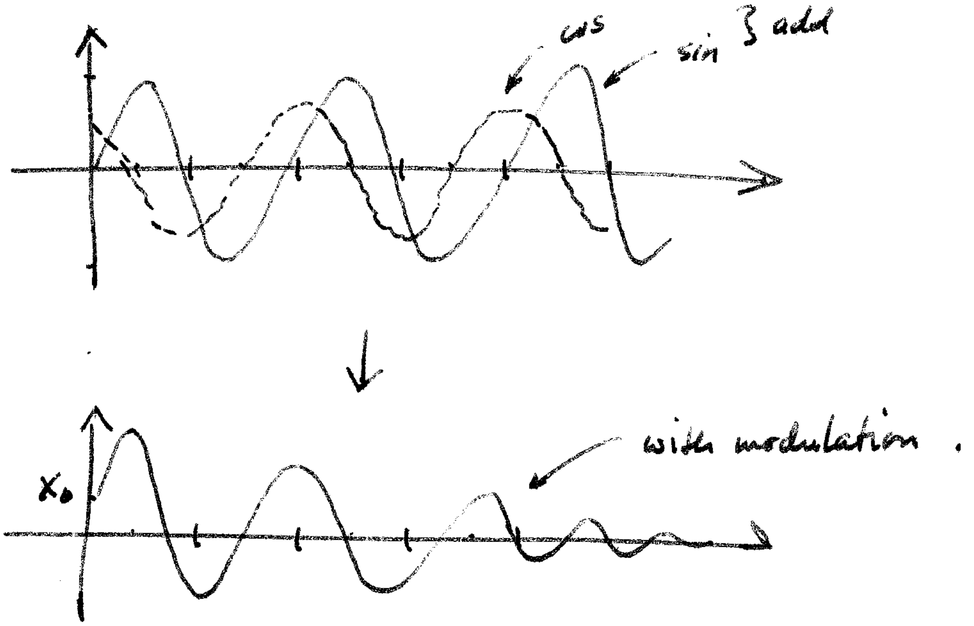
$$= \frac{x_0 e^{-\gamma t}}{2i\sqrt{\omega^2-\gamma^2}}\left[(\gamma + i\sqrt{\omega^2-\gamma^2})e^{i\sqrt{\omega^2-\gamma^2}\,t} + (-\gamma + i\sqrt{\omega^2-\gamma^2})e^{-i\sqrt{\omega^2-\gamma^2}\,t}\right]$$

$$= \frac{x_0 e^{-\gamma t}}{2i\sqrt{\omega^2-\gamma^2}}\left[\gamma\left(e^{i\sqrt{\omega^2-\gamma^2}\,t} - e^{-i\sqrt{\omega^2-\gamma^2}\,t}\right) + i\sqrt{\omega^2-\gamma^2}\left(e^{i\sqrt{\omega^2-\gamma^2}\,t} + e^{-i\sqrt{\omega^2-\gamma^2}\,t}\right)\right]$$

$$\therefore \boxed{x(t) = x_0 e^{-\gamma t}\left[\frac{\gamma}{\sqrt{\omega^2-\gamma^2}}\sin\left(\sqrt{\omega^2-\gamma^2}\,t\right) + \cos\left(\sqrt{\omega^2-\gamma^2}\,t\right)\right]}$$

$\longrightarrow$

$\therefore$ x is the sum of sin and cos; the amplitude of sin part may be greater or smaller than the cos part, depending on the precise value of ω/γ. The whole thing is then modulated (muted, in this case) with an exponentially decaying function $e^{-\gamma t}$, starting from x_0:



(iv) the main difference from that in the undamped case is the decay modulation of the periodic sol'n. Also, the frequency is reduced, since $\sqrt{\omega^2-\gamma^2} < \omega$. Hence, the oscillation period, $2\pi/\sqrt{\omega^2-\gamma^2}$, in the damped case is longer.

4 $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$

(i) strong damping $\Rightarrow \gamma^2 > \omega^2$. Following the same procedure in prob 3, get $\lambda_\pm = -\gamma \pm \sqrt{\gamma^2-\omega^2} \equiv -\gamma \pm \delta$; $\delta \equiv \sqrt{\gamma^2-\omega^2}$.
(Note, we could have used a similar definition in prob 3 to save us a bit of writing!)
$\therefore$ gen sol'n is: $x(t) = Ae^{-\gamma t}e^{\delta t} + Be^{-\gamma t} \cdot e^{-\delta t}$.

(ii) Since $\delta \in \mathbb{R}$, the two part (terms) in x are two decaying functions: $e^{-(\gamma-\delta)t}$ and $e^{-(\gamma+\delta)t}$; $\delta < \gamma$.

(iii) critically damped $\Rightarrow \gamma^2 = \omega^2 \Rightarrow \lambda = -\gamma$ (degeneracy!)
In this case, the gen. sol'n must be of the form,

$$\boxed{x(t) = (At + B)e^{-\gamma t}}$$

sol'n check: $\dot{x} = -\gamma(At+B)e^{-\gamma t} + Ae^{-\gamma t}$

$$\ddot{x} = \gamma^2(At+B)e^{-\gamma t} - \gamma Ae^{-\gamma t} - \gamma Ae^{-\gamma t}$$

$$\Rightarrow \gamma^2(At+B)e^{-\gamma t} - 2\gamma Ae^{-\gamma t} + 2\gamma Ae^{-\gamma t} - 2\gamma^2(At+B)e^{-\gamma t} + \omega^2(At+B)e^{-\gamma t} = (\omega^2-\gamma^2)(At+B)e^{-\gamma t} = 0,$$

when $\omega^2 = \gamma^2$ ✓.

(iv) let $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = [D^2 + 2\gamma D + \omega^2]x = 0$

$$\Rightarrow (D+\gamma+\delta)(D+\gamma-\delta) = D^2 + D\gamma - \cancel{D\delta} + \gamma D + \gamma^2 - \cancel{\gamma\delta} + \delta D + \cancel{\delta\gamma} - \delta^2$$

$$= D^2 + 2\gamma D + \gamma^2 - \delta^2 = D^2 + 2\gamma D + \omega^2.$$

But, since $\gamma^2 = \omega^2$, $\delta = 0 \Rightarrow [D^2 + 2\gamma D + \omega^2]x = (D+\gamma)(D+\gamma)x = 0$.
Now, let $(D+\gamma)x = y \Rightarrow (D+\gamma)y = 0 \Rightarrow Dy = -\gamma y$

$$\Rightarrow y = y_0 e^{-\gamma t} \Rightarrow (D+\gamma)x = y_0 e^{-\gamma t} \Rightarrow e^{\gamma t}(D+\gamma)x = y_0 = D(xe^{\gamma t})$$

IMPORTANT TRICK !!

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n.b., $D(xe^{\gamma t}) = (Dx)e^{\gamma t} + \gamma x e^{\gamma t} = (Dx + \gamma x)e^{\gamma t}$

$= e^{\gamma t}(D+\gamma)x$ ✓.

$\Rightarrow y_0 t + c = x e^{\gamma t}$, integrating $D(xe^{\gamma t}) = y_0$.

y_0, c are constants of integration. We can replace (or call) them A, B respectively.

$\therefore \; At + B = xe^{\gamma t} \Rightarrow \boxed{x(t) = (At+B)e^{-\gamma t}}$ ☒.