MTH5106 – Dynamics of Physical Systems **Dr. J. Cho**

Problem Set 5 (4 questions) **Due Tuesday, 24 Nov 2009, at 12:00 in the Lec Room**

Your solutions should be handed in *before* start of the lecture on the due date. (N.B., all of the problems in this CW simply review elementary calculus, linear algebra or ordinary differential equations – and hence can be answered without any knowledge of 'physics' or 'mechanics'.)

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1. For $x, \in \mathbb{R}$, $f(x) = e^x$ can be expanded in a Taylor series,

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\, \frac{x^n}{n!},$$

where $f^{(n)}(0)$ is the n-th derivatives of f evaluated at 0. For $\theta \in \mathbb{R}$ as well, $f(\theta) = \sin\theta$ and $f(\theta) = \cos\theta$ can be expanded in the same way.

(a) By letting $x = i\theta$, where $i = \sqrt{-1}$ and $\theta \in \mathbb{R}$, show that $e^x = \cos\theta + i\sin\theta$.

(b) Using the result from part (a), obtain the expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of e^x and e^{-x}.

(c) Using the result from part (b), show that $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$.

2. Given $C_i \in \mathbb{C}$ are constants $\forall\, i$, show that the following are equivalent to $C_1 \sin(\omega t + C_2)$:

(a) $C_3 \sin\omega t + C_4 \cos\omega t$

(b) $C_5\, e^{i\omega t} + C_6\, e^{-i\omega t}$

(c) $Re\,\{C_7\, e^{i(\omega t + C_8)}\}$ (i.e., "real part of" $\{\cdot\}$)

3. In two-dimensional space, the unit vectors of polar coordinates $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ is obtained as a simple (clockwise) rotation of the ordinary Cartesian coordinate unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ by an angle θ. This can be compactly written in the form,

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}.$$

Note that both coordinate systems are orthogonal. Show that the above is a correct description of the transformation between the two coordinate systems by describing a point P in both systems.

4. The displacement $x(t)$ from the equilibrium position of a simple harmonic oscillator (SHO) is governed by the ordinary differential equation,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x \;=\; 0,$$

where ω is a positive constant (angular frequency of the oscillation).

(a) Obtain and solve the characteristic equation. What is the general solution to the equation?

(b) Show by direct substitution that the answer obtained in part (a) is a solution to the SHO equation.

(c) If the initial positive displacement of the oscillator is $x(0) = X_0$, with velocity $\mathrm{d}x/\mathrm{d}t\,(0) = 0$, what is the full solution subject to this initial condition?

(d) Sketch the solution found in part (c).