MTH5106, DPS CW5 SOL'N

J. Cho

$\boxed{1}$

(a) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$

$\Rightarrow e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{6}\theta^3 + \frac{1}{24}\theta^4 + \frac{i}{120}\theta^5 + \dots$

Now, $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 - \dots$

and $\sin\theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \dots$

$\Rightarrow i\sin\theta = i\theta - \frac{i}{6}\theta^3 + \frac{i}{120}\theta^5 - \dots$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$ $\boxtimes$.

(b) $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta$, since $e^{-i\theta} = e^{i(-\theta)}$

$\therefore 2\cos\theta = e^{i\theta} + e^{-i\theta} \Rightarrow \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$

$2i\sin\theta = e^{i\theta} - e^{-i\theta} \Rightarrow \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \boxed{\frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}}$

(c) $\sin(\theta_1 + \theta_2) \overset{?}{=} \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1$

$$\Rightarrow \frac{e^{i(\theta_1+\theta_2)} - e^{-i(\theta_1+\theta_2)}}{2i} = \frac{e^{i\theta_1} e^{i\theta_2} - e^{-i\theta_1} e^{-i\theta_2}}{2i}$$

$$= \frac{e^{i\theta_1} \cdot e^{i\theta_2} - e^{-i\theta_1} \cdot e^{-i\theta_2} - e^{-i\theta_1} \cdot e^{i\theta_2} + e^{i\theta_1} \cdot e^{-i\theta_2}}{2 \cdot 2i} + \frac{e^{i\theta_1} \cdot e^{i\theta_2} - e^{-i\theta_1} \cdot e^{-i\theta_2} - e^{i\theta_1} \cdot e^{-i\theta_2} + e^{-i\theta_1} \cdot e^{i\theta_2}}{2 \cdot 2i}$$

$$\Rightarrow \frac{e^{i\theta_1} - e^{-i\theta_1}}{2i} \cdot \frac{e^{i\theta_2} + e^{-i\theta_2}}{2} + \frac{e^{i\theta_2} - e^{-i\theta_2}}{2i} \cdot \frac{e^{i\theta_1} + e^{-i\theta_1}}{2} = \sin\theta_1 \cdot \cos\theta_2 + \sin\theta_2 \cdot \cos\theta_1$$ $\boxtimes$.

$\boxed{2}$

(a) $c_1 \sin(\omega t + c_2) = c_1(\sin\omega t \cos c_2 + \sin c_2 \cos\omega t)$

$= (c_1 \cos c_2)\sin\omega t + (c_1 \sin c_2)\cos\omega t$

$= c_3 \sin\omega t + c_4 \cos\omega t \quad \boxtimes.$

(b) $c_1 \sin(\omega t + c_2) = c_1\left[\dfrac{e^{i(\omega t + c_2)} - e^{-i(\omega t + c_2)}}{2i}\right]$

$= \dfrac{c_1}{2i}\left(e^{i\omega t} \cdot e^{ic_2} - e^{-i\omega t} \cdot e^{-ic_2}\right)$

$= \left(\dfrac{c_1 e^{ic_2}}{2i}\right) e^{i\omega t} + \left(\dfrac{-c_1 e^{-ic_2}}{2i}\right) e^{-i\omega t}$

$= c_5 e^{i\omega t} + c_6 e^{-i\omega t} \quad \boxtimes.$

(c) Let $\phi = \omega t + c_8$ and $c = c_7$.

Then, $Re\{c_7 e^{i(\omega t + c_8)}\} = Re\{(c_r + ic_i)(\cos\phi + i\sin\phi)\}$

$= c_r \cos\phi - c_i \sin\phi = c_r(\cos\omega t \cdot \cos c_8 - \sin\omega t \cdot \sin c_8)$

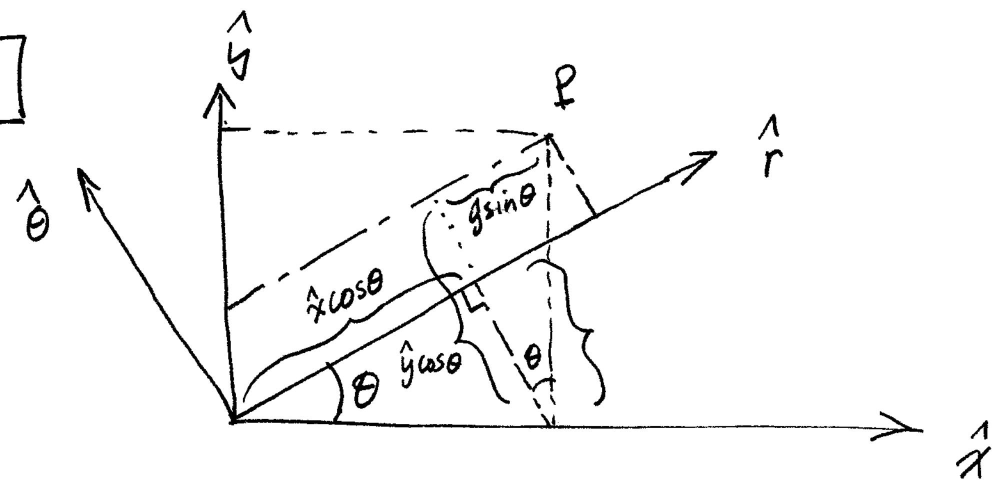
$- c_i(\sin\omega t \cdot \cos c_8 + \sin c_8 \cdot \cos\omega t)$

$= (c_r \cos c_8 - c_i \sin c_8)\cos\omega t + [-(c_r \sin c_8 + c_i \cos c_8)]\sin\omega t$

$= \sin\omega t\, c_1 \cos c_2 + \cos\omega t\, c_1 \sin c_2 = c_1 \sin(\omega t + c_2) \quad \boxtimes.$

$$\left.\begin{aligned} c_1 \cos c_2 &= -(c_r \sin c_8 + c_i \cos c_8) \\ \text{and } c_1 \sin c_2 &= (c_r \cos c_8 - c_i \sin c_8) \end{aligned}\right\} \tan c_2 = \frac{c_i \sin c_8 - c_r \cos c_8}{c_i \cos c_8 - c_r \sin c_8}$$

3



$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = \hat{y}\cos\theta - \hat{x}\sin\theta = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

or $$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$ ⊠.

---

4 $\quad \dfrac{d^2x}{dt^2} + \omega^2 x = 0 \quad ; \quad x = x(t)$

(a) characteristic eq'n: $m^2 + \omega^2 = 0 \Rightarrow \boxed{m = \pm i\omega}$

$\Rightarrow$ general sol'n is: $\boxed{x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}}$

(b) $\dfrac{dx}{dt} = \omega i c_1 e^{i\omega t} - i\omega c_2 e^{-i\omega t} \Rightarrow \dfrac{d^2x}{dt^2} = -\omega^2 c_1 e^{i\omega t} - \omega^2 c_2 e^{-i\omega t}$

$= -\omega^2 x \Rightarrow \boxed{\dfrac{d^2x}{dt^2} + \omega^2 x = 0}$

⊠.

(c) $x(0) = X_0 = c_1 + c_2$

and $\dfrac{dx}{dt}(0) = 0 = c_1 - c_2 \Rightarrow \boxed{c_1 = c_2} = \dfrac{X_0}{2}$

$\Rightarrow x(t) = \dfrac{X_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = \boxed{X_0 \cos\omega t}$

continued.

(d)