MTH5106 CW4 SOLUTIONS

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1 (i) $\vec{F} = m\vec{a} = m\dfrac{d\vec{v}}{dt} \Rightarrow m\vec{v}\cdot\dfrac{d\vec{v}}{dt} = \vec{F}\cdot\vec{v} = \vec{F}\cdot\dfrac{d\vec{r}}{dt}$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^{\,2}\right) = \frac{d}{dt}(KE) = \vec{F}\cdot\frac{d\vec{r}}{dt} \Rightarrow \frac{d(KE)}{dt}dt = \vec{F}\cdot\frac{d\vec{r}}{dt}dt$$

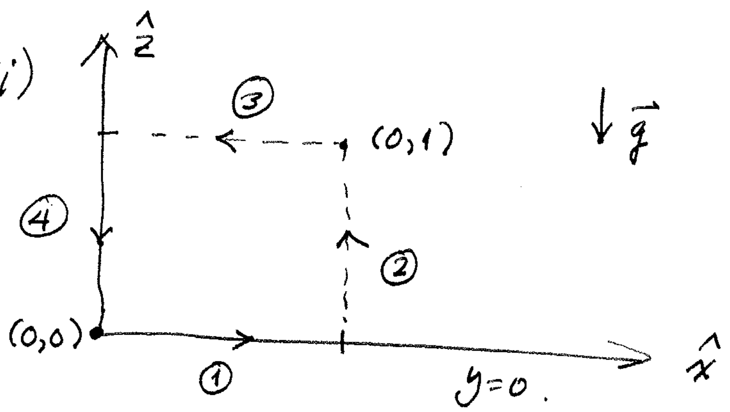
$\Rightarrow d(KE) = \vec{F}\cdot d\vec{r} \equiv dW$; dW is infinitesimal work done on the object.

$$\Rightarrow \int_1^2 d(KE) = \boxed{\int_1^2 dW = KE_2 - KE_1} \quad \boxtimes.$$

(ii) $$\Delta KE = KE_2 - \cancelto{0}{KE_1} = \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 -mg\hat{z}\cdot d\vec{r} = -mg\int_1^2 dz$$

$$= -mg(z_2 - z_1) = -mg\,\Delta z \quad \boxtimes.$$

(iii)

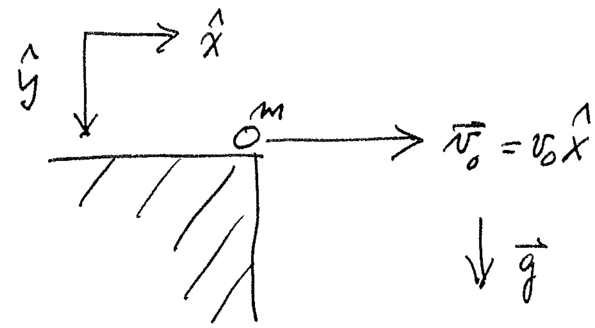


$$I = \oint \vec{F}\cdot d\vec{r} = \int_{①} \cancelto{0}{\vec{F}\cdot d\vec{x}} + \int_{②} \vec{F}\cdot d\vec{z} + \int_{③} \cancelto{0}{\vec{F}\cdot d\vec{x}} + \int_{④} \vec{F}\cdot d\vec{z},$$

since $\vec{F} = -mg\hat{z}$ and $d\vec{x} = dx\,\hat{x}$ (i.e., $\hat{x}\perp\hat{z}$).

$$\therefore I = -mg\int_0^1 dz - mg\int_1^0 dz = \boxed{0} \quad \boxtimes.$$

$\boxed{2}$



(i) $-\alpha m v^2$ has the unit of force – i.e., $\frac{kg\, m}{s^2}$.

$\therefore\ [\alpha]\, \cancel{kg} \cdot \frac{m^{\cancel{2}}}{\cancel{s^2}} = \cancel{kg}\frac{\cancel{m}}{s^2} \Rightarrow \boxed{[\alpha] = \frac{1}{m}}$

(ii) $F_x = m a_x \Rightarrow m\frac{dv_x}{dt} = -\alpha m v_x^2 \Rightarrow \boxed{\frac{dv_x}{dt} = -\alpha v_x^2}$

$$\Rightarrow \int_{v_0}^{v_x(t)} \tilde{v}_x^{-2}\, d\tilde{v}_x = \int_0^t -\alpha\, d\tilde{t} = -\alpha t = -\tilde{v}_x^{-1}\Big|_{v_0}^{v_x(t)} = \frac{1}{v_0} - \frac{1}{v_x(t)}$$

$$\Rightarrow \frac{1}{v_x(t)} = \frac{1}{v_0} + \alpha t = \frac{1 + v_0\alpha t}{v_0} \Rightarrow \boxed{v_x(t) = \frac{v_0}{1 + v_0\alpha t}}$$

(iii) $F_y = m a_y \Rightarrow m\frac{dv_y}{dt} = -\alpha m v_y^2 + mg \Rightarrow \boxed{\frac{dv_y}{dt} = -\alpha v_y^2 + g}$

$$\Rightarrow \int_0^{v_y(t)} \frac{d\tilde{v}_y}{g - \alpha\tilde{v}_y^2} = \int_0^t d\tilde{t} = t = \frac{\tanh^{-1}\left(v_y\sqrt{\frac{\alpha}{g}}\right)}{\sqrt{\alpha g}}$$, using the hint.

$$\Rightarrow t\sqrt{\alpha g} = \tanh^{-1}\left(v_y\sqrt{\frac{\alpha}{g}}\right) \Rightarrow \tanh(\sqrt{\alpha g}\, t) = v_y\sqrt{\frac{\alpha}{g}}$$

$$\Rightarrow \boxed{v_y(t) = \sqrt{\frac{g}{\alpha}} \cdot \tanh(\sqrt{\alpha g}\, t) = \sqrt{\frac{g}{\alpha}}\ \frac{e^{\sqrt{\alpha g}\, t} - e^{-\sqrt{\alpha g}\, t}}{e^{\sqrt{\alpha g}\, t} + e^{-\sqrt{\alpha g}\, t}}}$$

I.C. Check: $v_y(0) = \sqrt{g/\alpha}\,(1-1) = 0$ ✓.

(iv)

$$\sqrt{\alpha g}\, t \ll 1 \Rightarrow v_y \approx \sqrt{\frac{g}{\alpha}}\left(\frac{\cancel{1} + \sqrt{\alpha g}\, t - \cancel{1} + \sqrt{\alpha g}\, t}{1 + \cancel{\sqrt{\alpha g}\, t} + 1 - \cancel{\sqrt{\alpha g}\, t}}\right)$$

$$= \sqrt{\frac{g}{\alpha}} \cdot \frac{\cancel{2}\sqrt{\alpha g}\, t}{\cancel{2}} = \boxed{gt}$$

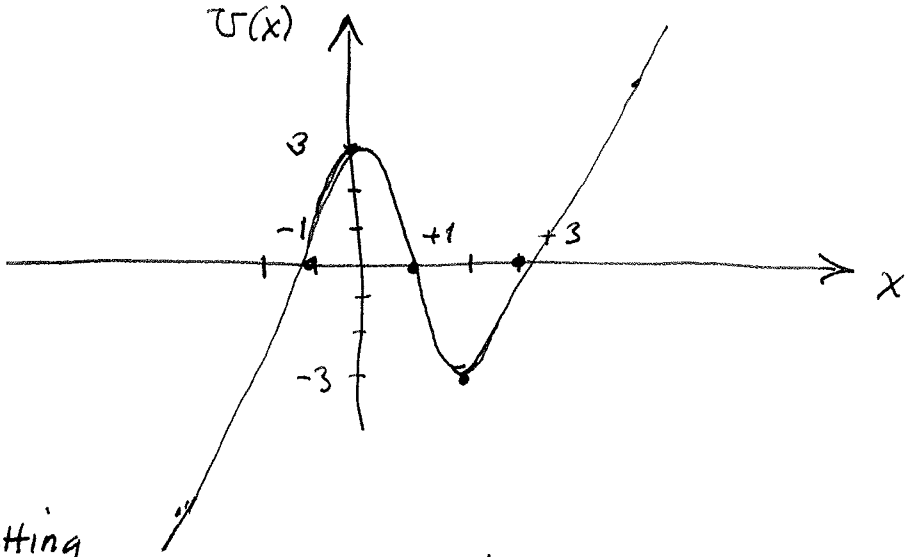
i.e., vel increases linearly with time early on.

For $t \to \infty$, we have $v_y \approx \sqrt{\frac{g}{\alpha}} \cdot \left(\frac{1-0}{1-0}\right) = \boxed{\sqrt{\frac{g}{\alpha}}}$

i.e., linear, "Terminal" velocity

$\boxed{3}$ $U(x) = x^3 - 3x^2 - x + 3$

(i) $U(x) = x^2(x-3) - (x-3) = (x^2-1)\cdot(x-3)$



(ii) Setting $\frac{dU}{dx} = 0$ gives the equlibrium pts $\forall x$ that satisfy the condition:

$$3x^2 - 6x - 1 = 0 \Rightarrow x_{eq} = \frac{6 \pm \sqrt{36+12}}{6} = 1 \pm \frac{1}{6}\sqrt{48}$$

$$\boxed{x_{eq} = 1 \pm \frac{2}{3}\sqrt{3}}$$

(iii) There are 2 eq. points. To see whether the pt is stable/unstable, check the 2nd-derivative of $U(x)$ at the pt:

$$\frac{d^2U}{dx^2} = 6x - 6 \quad \text{(n.b., do not set this to zero. Why?).}$$

$x_{eq}^1 = 1 + \frac{2}{3}\sqrt{3} \Rightarrow \left.\frac{d^2U}{dx^2}\right|_{x_{eq}^1} = 6(1+\frac{2}{3}\sqrt{3}) - 6 > 0$. $\therefore$ STABLE

$x_{eq}^2 = 1 - \frac{2}{3}\sqrt{3} \Rightarrow \left.\frac{d^2U}{dx^2}\right|_{x_{eq}^2} = 6(1-\frac{2}{3}\sqrt{3}) - 6 < 0$. $\therefore$ UNSTABLE

→ continued.

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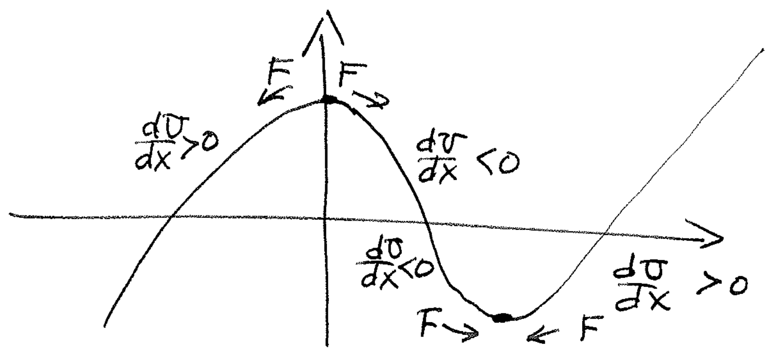

Force $= -\frac{dU}{dx}$. $\therefore$ Force around the eq. pts is as indicated (and, as expected).

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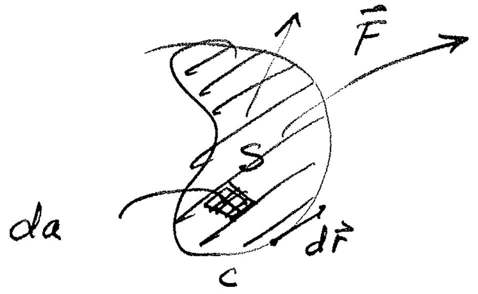
4 (i) Given a contour loop C enclosing an area S, oriented with the normal $\hat{n}$, the curl of a vector field $\vec{F}$ threading S (in the $\hat{n}$-direction) is equal to the line (contour) integral of $\vec{F}$ around c (taken counter-clockwise):



$$\int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,da = \oint_C \vec{F}\cdot d\vec{r}$$

(ii) a) The work done by the field is independent of the path taken by a particle moving in the field (i.e., the work depends only on the end pts.).

b) The (force) field can be written as a gradient of a scalar function, called "potential", traditionally written: $\vec{F} = -\vec{\nabla} U$

c) $\vec{\nabla}\times\vec{F} = 0$

(n.b., the three actually mean the same thing).

$\longrightarrow$ continued.

(iii) $\vec{F}(\vec{r}) = -\frac{k}{r^2}\hat{r}$ is given. We can show that $\vec{F}$ is conservative in several different ways:

a) Note that $\vec{F}$ is radial ("central"). Hence, $\vec{\nabla}\times\vec{F}=0$.

b) We can also show a) directly. Rewrite $\vec{F}$:

$$\vec{F}(\vec{r}(x,y,z)) = \vec{F}(x,y,z) = -\frac{k}{r^3}\vec{r}, \text{ using } \vec{r}=r\hat{r}$$

$$= \frac{-k}{(x^2+y^2+z^2)^{3/2}}\cdot(x,y,z), \text{ using } r^2=x^2+y^2+z^2 \text{ and } \vec{r}=(x,y,z).$$

Now, $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-kx}{(x^2+y^2+z^2)^{3/2}} & \frac{-ky}{(x^2+y^2+z^2)^{3/2}} & \frac{-kz}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= -k\hat{i}\left[z\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2} - y\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-3/2}\right]+\dots$$

$$= -k\hat{i}\left[-\frac{3}{2}\cdot\frac{2zy}{(\quad)^{5/2}} + \frac{3}{2}\cdot\frac{2yz}{(\quad)^{5/2}}\right]+\dots$$

$$= 0\hat{i}+0\hat{j}+0\hat{k} = \boxed{0}$$ ☒.

c) We can also show that $\exists\ U(r) \ni \vec{F}=-\vec{\nabla}U$:

$$\boxed{U(r) = -\frac{k}{r}} \quad \underset{\text{check}}{\Rightarrow} -\vec{\nabla}U = -\frac{\partial U}{\partial r}\hat{r} \text{ (in spherical coordinates)}$$

$$= -\frac{k}{r^2}\hat{r} = \vec{F}$$ ☒.