MTH5106, DPS' 2008 MIDTERM SOL'NS

$\boxed{1}$ (a)

$$\vec{A} = 3\hat{i} + \hat{j}, \quad \vec{B} = \hat{i} + 2\hat{j}$$

$$\text{area} = |\vec{A} \times \vec{B}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = |(6-1)\hat{k}| = \boxed{5}$$

(b) $\vec{C} = x^2 y\,\hat{i} + z\hat{j} + 3\hat{k}$

$$\vec{\nabla} \cdot \vec{\nabla} \vec{C} = \nabla^2 \vec{C} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{C} = \frac{\partial^2}{\partial x^2}\vec{C} + \frac{\partial^2}{\partial y^2}\vec{C} + \frac{\partial^2}{\partial z^2}\vec{C}$$

$$= \frac{\partial}{\partial x}(2xy\hat{i}) + \frac{\partial}{\partial y}(x^2\hat{i}) + \frac{\partial}{\partial z}(\hat{j})$$

$$= \boxed{2y\hat{i}}$$

$\boxed{2}$ (a) $v = v_0 + at = \dfrac{ds}{dt} \Rightarrow s - s_0 = v_0 t + \frac{1}{2}at^2$

$$\Rightarrow \boxed{s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2} \quad \boxtimes$$

(b)

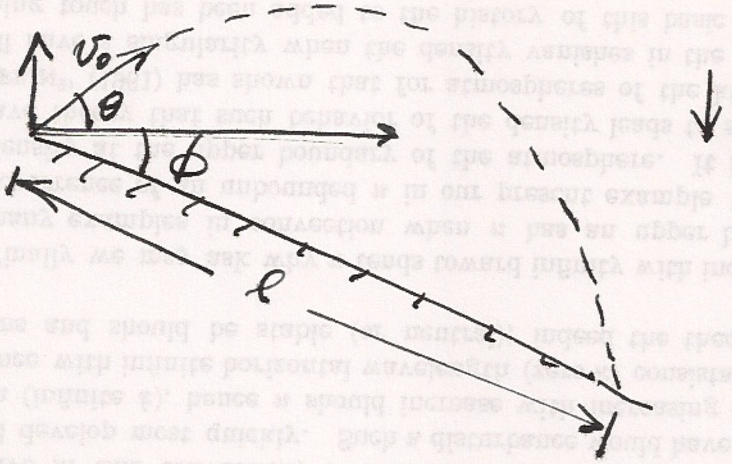
$$s - s_0 = v_0 t + \frac{1}{2}at^2 = v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$$

$$= \frac{v_0 v}{a} - \frac{v_0^2}{a} + \frac{1}{2a}\left(v^2 - 2vv_0 + v_0^2\right)$$

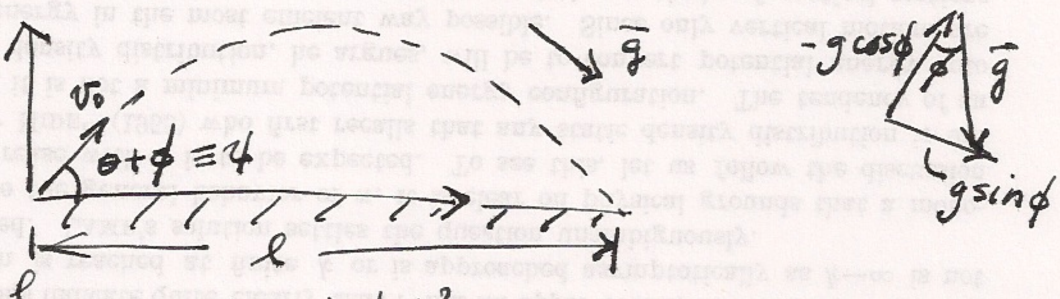
$$= -\frac{v_0^2}{2a} + \frac{v^2}{2a} \Rightarrow v^2 = v_0^2 + 2a(s - s_0)$$

$$\Rightarrow \boxed{v(t) = \left[v_0^2 + 2a(s - s_0)\right]^{1/2}} \quad \boxtimes.$$

3 (a)



(b)



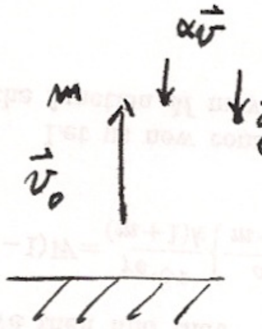
$$l = x = v_{0x} t + \frac{1}{2} g \sin\phi \, t^2$$

$$v_y^{0} = v_{0y} - g\cos\phi \, t_u \Rightarrow t = \frac{2 v_{0y}}{g\cos\phi}$$

$$\Rightarrow l = v_0 \cos\psi \, \frac{2 v_{0y}}{g\cos\phi} + \frac{g\sin\phi}{2} \cdot \frac{4 v_{0y}^2}{g^2 \cos^2\phi}$$

$$= \frac{2 v_0^2 \sin\psi\cos\psi}{g\cos\phi} + \frac{2 v_0^2 \sin^2\psi \tan\phi}{g\cos\phi}$$

$$\therefore \boxed{l = \frac{2 v_0^2}{g} \cdot \frac{\sin\psi}{\cos\phi}\left(\cos\psi + \sin\psi \tan\phi\right)} \ ; \ \psi = \theta + \phi$$



(a) $[\alpha\vec{v}] = \frac{N}{kg} = \frac{kg\ m/s^2}{kg} = m/s^2 \Rightarrow \boxed{[\alpha] = 1/s}$

(also, $e^{-\alpha t} \Rightarrow [\alpha] = 1/s$)

(b) $m\frac{dv}{dt} = -\alpha m v - mg \Rightarrow \boxed{\frac{dv}{dt} = -(\alpha v + g)}$

$$\frac{dv}{\alpha v + g} = -dt \Rightarrow \frac{1}{\alpha}\ln(\alpha v + g)\Big|_{v_0}^{v} = -t$$

$$\Rightarrow e^{-\alpha t} = \frac{\alpha v(t) + g}{\alpha v_0 + g}$$

$$\Rightarrow (\alpha v_0 + g)e^{-\alpha t} - g = \alpha v(t)$$

$$= \alpha v_0 e^{-\alpha t} - g(1 - e^{-\alpha t})$$

$$\Rightarrow \boxed{v(t) = v_0 e^{-\alpha t} - \frac{g}{\alpha}(1 - e^{-\alpha t})}$$

I.C. Check: $t=0 \Rightarrow v = v_0$ ✓

(c) when $\alpha = 0$, $\boxed{u(t) = v_0 - gt}$

when $\alpha t \ll 1$, $v(t) \approx v_0(1 - \alpha t) - \frac{g}{\alpha}(1 - 1 + \alpha t)$

$$= v_0 - v_0\alpha t - gt = u(t) - v_0\alpha t$$

$$\Rightarrow \boxed{u(t) - v(t) = v_0 \alpha t}$$ ☒

(d) when $\alpha = 0$, $\boxed{t = \frac{v_0}{g}}$ is the time the stone stops moving upward. When $\alpha \neq 0$ and $v = 0$, then

$$e^{-\alpha t} = \frac{g}{\alpha v_0 + g} \Rightarrow e^{\alpha t} = \frac{\alpha v_0 + g}{g} = \frac{\alpha v_0}{g} + 1 \Rightarrow \alpha t = \ln\left(1 + \frac{\alpha v_0}{g}\right)$$

When $\frac{\alpha v_0}{g} \ll 1$, then $\ln\left(1 + \frac{\alpha v_0}{g}\right) \approx \frac{\alpha v_0}{g} = \alpha t \Rightarrow \boxed{t = \frac{v_0}{g}}$ same as when $\alpha = 0$.