Statistics of Lagrangian quantum turbulence

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We consider the dynamics of small tracer particles in turbulent quantum fluids. The complicated interaction processes of vortex filaments, the quantum constraints on vorticity, and the varying influence of both the superfluid and the normal fluid on the tracer particle effectively lead to a superstatistical Langevin-like model that in a certain approximation can be solved analytically. An analytic expression for the probability density function of velocity v of the tracer particle is derived that exhibits not only the experimentally observed v^{-3} tails but also the correct behavior near the center of the distribution, in excellent agreement with experimental measurements and numerical simulations. Our results are universal and do not depend on details of the quantum fluid.

Quantum turbulence is a phenomenon of utmost interest in current fluid mechanics research [1–10]. The turbulent behavior of a quantum liquid such as ^{4}He is very different from classical turbulence since vortices are quantized. This means the circulation cannot take arbitrary values as in classical turbulence, and there is also no viscous diffusion of vorticity as in classical turbulence. Recent measurements [1] and simulations [2,3] have shown that this has profound influence on various measurable observables, most notably the velocity distributions of small test particles embedded in the turbulent flow. Whereas for classical turbulence there is near-Gaussian behavior [11–13], for quantum turbulence one typically observes power-law tails in the high-velocity region [1–3,14].

The velocity statistics has been subject of several recent papers. Paoletti *et al.* [1] conducted a seminal experiment using solid hydrogen tracers in turbulent superfluid ^{4}He and found that the distribution of velocity components v_i of the tracer particles exhibits a power-law distribution $p(v_i) \propto v_i^{-3}$ for large values of v_i. Similar results were confirmed by White *et al.* [2]. They performed numerical simulations of quantum turbulence in a trapped Bose-Einstein condensate by calculating solutions of the Gross-Pitaevski equation. The associated probability density function (PDF) of each velocity component v_i ($i = x,y,z$) was computed directly and it was confirmed that the velocity statistics is non-Gaussian and obeys a power-law distribution $p(v_i) \propto v_i^{-b}$ with $3.3 < b < 3.6$. In the following, for ease of notation, we often suppress the index i. Adachi *et al.* [3] numerically computed the velocity field of a superflow by calculating the Biot-Savart velocity induced by vortex filaments in steady counterflow turbulence. They found that the resulting PDF exhibits a near-Gaussian distribution in the low-velocity region whereas a power law $p(v) \propto v^{-3}$ is observed in the high-velocity region.

Velocity distributions with a power-law decay have been previously observed in two-dimensional *classical* turbulence [14–18]. But classical two-dimensional turbulence models, though extremely useful from a phenomenological point of view, cannot serve as a proper explanation of the observed power laws in quantum turbulence, and other approaches are needed. What is indeed missing so far is a theory of *Lagrangian quantum turbulence*. Whereas three-dimensional Lagrangian turbulence is a well-established subject area for classical

turbulence [19–27], very little is known for the quantum case.

In this Rapid Communication we will introduce a simple but powerful dynamical model of the dynamics of a tracer particle embedded in a quantum liquid. This model will be based on a superstatistical stochastic differential equation. The superstatistics concept, introduced in Ref. [28], has proved to be a very powerful method for modeling a variety of complex systems [29–32], including driven nonequilibrium situations and classical hydrodynamic turbulence [24–27,33]. Here we apply this concept to quantum turbulence. The result is a dynamical theory that quite precisely reproduces the observed velocity statistics in quantum turbulence and that also allows for some analytic predictions. In particular, the power-law exponent -3 follows from our theoretical consideration in a natural way, and, moreover, a universal prediction for the entire shape of the velocity distribution is obtained, which is in excellent agreement with experimental measurements.

Let us denote the velocity of a Lagrangian tracer particle embedded in the quantum liquid by $\boldsymbol{v}(t)$. This tracer impurity can be, e.g., an electron, ion, or solid hydrogen. We start from a simple local dynamics which will later be extended to a superstatistical model. Consider a linear stochastic differential equation of the form

$$\dot{\boldsymbol{v}}(t) = -\Gamma \boldsymbol{v}(t) + \Sigma \boldsymbol{L}(t). \tag{1}$$

Here $\boldsymbol{L}(t)$ is a rapidly fluctuating stochastic process representing rapid forces in the quantum liquid on a fast time scale, and Γ and Σ are 3×3 matrices. The above equation simply says that locally a tracer particle is driven by chaotic forces $\boldsymbol{L}(t)$ from the turbulent flow and at the same time there are damping processes, described by Γ. Since the chaotic forces act rapidly, we approximate $\boldsymbol{L}(t)$ by Gaussian white noise. Γ and Σ are matrix-valued stochastic processes which evolve on a much larger time scale than $\boldsymbol{L}(t)$. The particle is driven by a mixture of normal and superfluid, and depending on which component dominates, the effective friction described by Γ will be very different.

In fact, a tracer particle in a quantum liquid is directly forced by viscous forces of the normal fluid component, whereas the superfluid flows right around the particle without exerting viscous forces. The superfluid component, however, can influence the particle by pressure forces, and these pressure

forces can produce trapping of the tracer particle on quantized vortices. A very complex dynamics arises and a test particle trapped on a quantized vortex may be driven to orbit another quantized vortex. For more details on the dynamics of quantum turbulence, we refer to Refs. [34–37].

A characteristic property of quantum turbulence is a spatiotemporally varying vorticity field represented by one-dimensional topological vortices that reconnect and merge at random moments of time. A tracer particle will rotate for a short around a local unit vector $\boldsymbol{e}$ whose direction will be a random variable, influenced by the nearby vortex filament structure in the quantum liquid. Hence, as a special case of Eq. (1) we may consider the local dynamics

$$\dot{\boldsymbol{v}} = -\gamma(t)\boldsymbol{v} + \omega[\boldsymbol{e}(t) \times \boldsymbol{v}] + \sigma(t)\boldsymbol{L}(t) \qquad (2)$$

(see also Ref. [19]). We assume that the effective damping constant γ and the noise strength σ are functions of t, and so is ω and the direction of $\boldsymbol{e}$. The second term on the right hand side of Eq. (2) represents the rotational movement of the particle around the vortex filament. The unit vector $\boldsymbol{e}$ and the noise strength σ evolve stochastically on a large time scale T_e and T_σ, respectively.

The basic idea of the superstatistics approach is to regard the *parameters* of a local stochastic differential equation as random variables as well [33]. This means both γ and ω can take on very different values during time evolution, and so can the direction of $\boldsymbol{e}$. A very small γ corresponds to nearly undamped motion for a limited amount of time. A very small ω corresponds to almost no rotation, i.e., straight movement for a limited amount of time. All these cases are included as possible local dynamics and averaged over in the superstatistical approach.

In a quantum turbulent flow, the superfluid component flows without dissipation while being subject to certain quantum mechanical constraints. These quantum restrictions imply that the typical form of rotational motion allowed in the superfluid component is in the form of a thin vortex line, whose circulation around its core is quantized rather than arbitrary as in classical fluids. The magnitude of the velocity field of the fluid particle at distance r from the core of the vortex filament is given by [3]

$$v = |\boldsymbol{v}| = \frac{\kappa}{2\pi r}, \qquad (3)$$

where $\kappa = \frac{h}{m} \approx 9.97 \times 10^{-4}$ cm^2/s is the quantum of circulation, h is Planck's constant, and m is the mass of the fluid atom, in our case helium.

For an ordinary spherical Brownian particle in a laminar viscous liquid one has constant damping due to Stokes' law:

$$\gamma = \frac{6\pi \nu \rho a}{M}. \qquad (4)$$

Here ν is the kinematic viscosity of the liquid, ρ is the fluid density, M is the mass, and a the radius of the tracer particle.

For quantum turbulence, the effective dissipation acting on the tracer particle is influenced by many competing effects, and it fluctuates strongly depending on whether or not the particle is close to a vortex filament. Far away from a vortex filament, the movement will be dominated by Brownian motion similar as in a normal liquid, whereas close to a vortex filament

the movement will be very rapid and almost friction free, dominated by the superfluid.

To take into account the fact that the effective friction in Eq. (2) is fluctuating, we may write quite generally

$$\gamma = \frac{1}{D^2} \nu \frac{1}{2} \sum_{i=1}^{n} X_i^2, \qquad (5)$$

where D is a characteristic length scale and the X_i are dimensionless random variables that evolve in time and space. We have squared the random variables because for physical reasons γ must always be positive, though values close to 0 are possible. n denotes the number of degrees of freedom that influence the fluctuating effective friction. Of course, the simplest model is to assume that the X_i are a rescaled sum of many microscopic random variables that act almost independently. Thus the central limit theorem suggests to assume that the X_i are Gaussian random variables.

The quantum mechanical constraint given in Eq. (3) tells us that the average rotational velocity of a tracer particle is very high near the vortex core (for small distance r). Therefore, the effective friction parameter γ acting on the tracer particle in Eq. (2) is small if the particle is very near to a vortex core. This means that $\sum_i^n X_i^2$ is small. On the other hand, if the test particle is very far from a vortex filament, then γ is large and the friction effects are strong, mainly due to the normal fluid component. This suggests the physical interpretation that the X_i's may just be identified with the perpendicular distances of the test particle from the nearest vortex filament. The vortex filaments themselves of course evolve in a highly complicated stochastic way. Since only the distance perpendicular to the nearest vortex filament is relevant, for a three-dimensional quantum liquid we have $n = 2$, that is, two degrees of freedom. The distance r of the test particle from the vortex filament becomes a random variable given by

$$r^2 = (X_1^2 + X_2^2)D^2, \qquad (6)$$

where again D is a suitable spatial scale introduced for dimensional reasons.

We may estimate this length scale D as follows: For large distances r, of the order of average vortex filament distance d in the turbulent flow, the tracer particle follows a nearly normal type of Brownian motion, with Stokes' law (4) valid in good approximation. Putting $r = \frac{d}{2}$ into Eqs. (6), (5), and (4), one arrives at the following estimate for the length scale D:

$$D = \left(\frac{Md^2}{48\pi\rho a}\right)^{\frac{1}{4}}. \qquad (7)$$

Clearly, our model requires small particles with $a \ll d$, if larger scales $a > d$ are probed, one just gets ordinary Brownian motion with Gaussian behavior [38,39].

The velocity distribution of the small tracer particle in the quantum turbulent flow described by Eq. (2) can now be calculated by using standard techniques of superstatistics [27]. We first assume, for simplicity, a constant γ and define the parameter $\beta := \frac{2\gamma}{\sigma^2}$, which in equilibrium statistical mechanics corresponds to the inverse temperature, whereas here it is more a measure of distance from the nearest vortex filament. On time scales t satisfying $\gamma^{-1} \ll t \ll T_\sigma$ the stationary distribution of the tracer particle described by Eq. (2) for fixed $\beta = \frac{2\gamma}{\sigma^2}$ is

given by the Gaussian distribution

$$p(v|\beta) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{1}{2}\beta v^2}, \tag{8}$$

assuming uniform distribution of the random vectors $\mathbf{e}$. The situation becomes different for fluctuating β, that is, if one allows the parameters γ (or σ) in Eq. (2) to be varying as well. Assuming that $\hat{X}_1, \ldots, \hat{X}_n$ are independent Gaussian random variables, the resulting distribution of $\beta = \sum_{i=1}^n \hat{X}_i^2$ is a χ^2 distribution of degree n, i.e.,

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{n}{2\beta_0}\right)^{\frac{n}{2}} \beta^{\frac{n}{2}-1} e^{-\frac{n\beta}{2\beta_0}}. \tag{9}$$

The average of the fluctuating β is given by

$$\langle \beta \rangle = n \langle \hat{X}_i^2 \rangle = \int_0^\infty \beta f(\beta) = \beta_0 \tag{10}$$

and the variance by

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{n}\beta_0^2. \tag{11}$$

The probability density to observe the velocity v of the test particle for any value of β is given by the marginal probability $p(v)$ as follows:

$$p(v) = \int_0^\infty f(\beta)p(v|\beta)d\beta. \tag{12}$$

Substituting $p(v|\beta)$ and $f(\beta)$ from Eqs. (8) and (9) into Eq. (12), we obtain, after a short calculation,

$$p(v) = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\beta_0}{\pi n}\right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{\beta_0}{n}v^2\right)^{\frac{n}{2}+\frac{1}{2}}}. \tag{13}$$

These types of distributions play an important role in q-generalized versions of statistical mechanics [40,41], with the entropic index q related to the parameter n by $q = 1 + \frac{2}{n+1}$.

As we mentioned earlier the velocity of the tracer particle depends on the perpendicular distance between the particle and the nearest evolving (and sometimes merging) vortex filament. Therefore, the relevant degrees of freedom are $n = 2$ for three-dimensional quantum turbulence. By substituting $n = 2$ in Eq. (13) one obtains

$$p(v) = \frac{\sqrt{\beta_0}}{(2 + \beta_0 v^2)^{\frac{3}{2}}}. \tag{14}$$

Clearly, for large v this implies power-law tails

$$p(v) \propto v^{-3}. \tag{15}$$

The remarkable result, however, is that we do not only get the power-law tails but a concrete prediction for the entire shape of the probability distribution, including the region near the maximum.

The probability distribution of kinetic energy E can be calculated from Eq. (13) by using a simple transformation of random variables. For a particle of unit mass $E = g(v) = \frac{1}{2}v^2$, hence $v = g^{-1}(E) = \sqrt{2E}$ and

$$p_E(E)dE = p_v(v)2dv, \tag{16}$$

the factor 2 coming from the fact that there are two solutions $\pm v$ for the same energy E. This leads to the probability distribution of energy

$$p_E(E) = 2p_v(g^{-1}(E)) \left| \frac{dg^{-1}(E)}{dE} \right|$$

$$= \sqrt{2} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\beta_0}{\pi n}\right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{2\beta_0}{n}E\right)^{\frac{n}{2}+\frac{1}{2}}} \frac{1}{\sqrt{E}}. \tag{17}$$

For $n = 2$ this predicts power-law tails proportional to E^{-2} for large E.

So far our model was based on a situation where the average velocity v of the particle is zero. Of course, in experiments there is often a drift velocity in the system that gives a nonzero mean velocity c to the test particle. In this case one has to replace v by $v - c$ in the model equations we derived so far, and for $n = 2$ one ends up with

$$p(v) = \frac{\sqrt{\beta_0}}{[2 + \beta_0(v-c)^2]^{\frac{3}{2}}}. \tag{18}$$

Let us now compare our model prediction with the experimental data obtained by Paoletti *et al.* [1]. Figure 1 shows the experimental data for both velocity components v_x and v_z, and a fit by our analytic formula. An excellent fit is obtained. It is remarkable that the fit is not only correctly producing the power-law tails but also the vicinity of the maximum. To illustrate this, Fig. 2 shows the same data in a linear plot.

Let us mention that our model directly predicts the power-law exponent -3 in a universal way. The value -3 is a consequence of the fact that vortex filaments are thin one-dimensional structures embedded into three-dimensional space, thus leading to $n = 2$ in Eq. (6). Our model also correctly reproduces the E^{-2} tails of the energy spectrum observed by Paoletti *et al.* [1].

Finally, we can also predict the value $\beta_0 \simeq 4.5$ of the variance parameter to be used in Eq. (18). To see this, let us recall that Paoletti *et al.*, in their experiment [1],
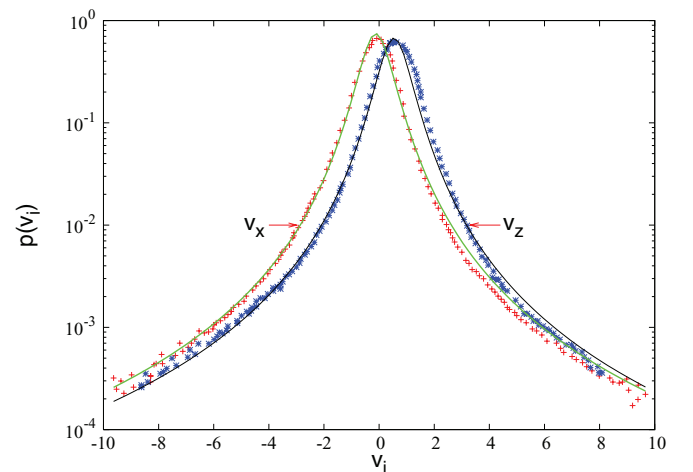

FIG. 1. (Color online) Experimental data of Paoletti *et al.* [1] for v_x and v_z (normalized by standard deviations). Also shown is a fit using Eq. (18) with $\beta_0 = 4.5$ and $c = -0.12$ for v_x, respectively, $c = 0.54$ for v_z (solid lines).
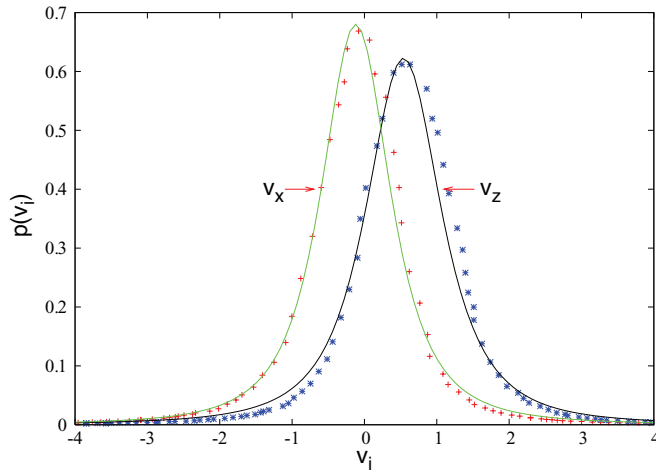
FIG. 2. (Color online) Same as Fig. 1 but a linear scale is chosen, which emphasizes the vicinity of the maximum. The variance parameter is slightly adjusted to obtain optimum coincidence in the central region.

rescaled their measured velocity data to variance 1. In these units their maximum velocity measured was $v_{max} \simeq 10$ (see Fig. 1). Strictly speaking, the variance does not exist for any distribution that decays as v^{-3} for large $|v|$, but what exists is of course the variance as calculated for a given experimental cutoff v_{max}. From

$$1 = \langle v^2 \rangle \simeq 2 \int_0^{v_{max}} p(v)v^2 dv \simeq \frac{2}{\beta_0} \log v_{max} \qquad (19)$$

we obtain the predicted value $\beta_0 \simeq 2 \log v_{max} \approx 4.6$, in agreement with what yields the optimum fit in Fig. 1. Thus, besides the (nonuniversal) systematic drift velocity c, all relevant parameters are predicted from first principles.

It is possible to extract further dynamical information out of our model. Of particular interest is, for example, the two-point correlation function $C(t) = \langle v(\tau)v(\tau + t) \rangle$ of the velocity components of the Lagrangian tracer particle. First, let us slightly generalize Eq. (2) by introducing a memory kernel so that a more general local dynamics reads

$$\dot{\boldsymbol{v}} = -\gamma \boldsymbol{v} + \alpha \int_0^t dt' e^{-\eta(t-t')} \boldsymbol{v} + \omega [\boldsymbol{e}(t) \times \boldsymbol{v}] + \sigma(t) \boldsymbol{L}(t). \qquad (20)$$

This model is then again extended in a superstatistical way by letting σ fluctuate. By employing techniques from nonequilibrium statistical mechanics as developed in Ref. [42] one can show that in this case the correlation function $C(t)$ decays as a sum of two exponentials,

$$C(t) \sim C_1 e^{-\gamma_+ t} + C_2 e^{-\gamma_- t}, \qquad (21)$$

whereas the prediction of the probability distribution Eq. (18) stays the same. For $\omega \to 0$ the constants $\gamma_{\pm}$ are given by

$$\gamma_{\pm} = \tfrac{1}{2}[\eta + \gamma \pm \sqrt{(\eta - \gamma)^2 + 4\alpha}]. \qquad (22)$$

For $\alpha = 0$ the correlation function just reduces to a single exponential with relaxation time γ^{-1}.

To conclude, in this Rapid Communication we have developed a superstatistical dynamical model of Lagrangian quantum turbulence. This model predicts that the velocity statistics of small tracer particles in a quantum turbulent flow asymptotically obey a power-law distribution $p(v) \propto v^{-3}$ and the distribution of energy follows a power law as well, i.e., $p(E) \propto E^{-2}$. These results are in excellent agreement with Paolleti *et al.*'s measurements [1] as well as with the numerical results obtained by other authors [2,3]. Our theory provides a universal prediction given by (18) for both the center and the tail parts of the velocity distribution. The underlying stochastic model arises quite naturally out of the fact that small tracer particles see fluctuating effective frictions, depending on the distance to the nearest vortex filament.

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