Comment on "Proposed central limit behavior in deterministic dynamical systems"

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In a recent Brief Report [Phys. Rev. E **79** (2009) 057201], Grassberger re-investigates probability densities of sums of iterates of the logistic map near the critical point and claims that his simulation results are inconsistent with previous results obtained by us [U. Tirnakli *et al.*, Phys. Rev. E **75** (2007) 040106(R) and Phys Rev. E **79** (2009) 056209]. In this comment we point out several errors in Grassberger's paper. We clarify that Grassberger's numerical simulations were mainly performed in a parameter region that was explicitly excluded in our 2009 paper and that his number of iterations is insufficient for the region chosen. We also show that, contrary to what is claimed by the author, (i) Lévy distributions are irrelevant for this problem, and that (ii) the probability distributions of sums that focus on transients are unlikely to be universal.

In a recent Brief Report [1], Grassberger claims that his numerical re-investigation of the probability density of sums of iterates of the logistic map near to the critical point of period doubling accumulation is inconsistent with our results previously obtained in [2, 3]. In [2] we provided for the first time numerical evidence for the possible relevance of q-Gaussians for this problem, and in [3] a more detailed investigation was performed with the main result that q-Gaussians are indeed a good approximation of the numerical data if the parameter distance to the critical point and the number of iterations entering the sum satisfy a scaling condition that was derived in [3]. In [1] the author also claims that a) Lévy distributions could give an equally good fit to the data as q-Gaussians b) Lévy statistics might have a better theoretical basis for this problem than q-statistics c) new types of distributions that he obtains by not neglecting transients could be universal.

In this note we point out that the paper [1] is misleading since most of the numerics performed in that paper operates in a parameter region that we explicitly excluded by the scaling condition derived in [3]. In the parameter region chosen by Grassberger, his statistics is insufficient in the sense that much larger numbers of iterates would be needed to observe q-Gaussian distributions. Moreover we show that claims a) and b) are incorrect and that there is no theoretical or numerical basis for claim c).

Let us use the same notation as in [1]. The object of study are sums Y of iterates x_i of the logistic map $f(x) = 1 - ax^2$ with parameter a close to the critical point $a_c = 1.4011551890920506...$ of period doubling accumulation. The sum consists of N iterates. One starts from an ensemble of uniform initial conditions and the first N_0 iterates are ommitted:

$$Y = \sum_{i=N_0+1}^{N_0+N} x_i \tag{1}$$

The question investigated in [1, 2, 3] is what probability distribution is to be expected for the random variable Y, since the ordinary Central Limit Theorem (CLT) is not valid close to the critical point due to strong correlations between the iterates.

Before discussing the results of [1], we first point out a few formal errors in [1]. If a is slightly above the critical a_c then it is well-known that the attractor of the logistic map consists of $n = 2^k$ chaotic bands. In [1] it is stated (1

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line below caption of Fig. 3) that $k \approx (a - a_c)^{1/\delta}$, where δ is the Feigenbaum constant. This statement is obviously wrong, the correct relation is

$$k = -\frac{\ln|a - a_c|}{\ln \delta}. \tag{2}$$

There are a few further formal errors in [1]. In the caption of Fig. 4 in [1] it is said that the figure shows numerical data for the probability density for various values of N and n. However, the actual data displayed in Fig. 4 seem to correspond to tuples of the form (n, N), i.e., the order of n and N has been swapped. Another error is the fact that the absolute value is missing when the author refers to the z-logistic map $f_{a,z}(x) = a - |x|^z$ three lines after eq.(2).

Let us now come to the actual content of the paper. In [3] we pointed out that the problem is more complex than the ordinary CLT, since two limits have to be performed simultaneously: $a \to a_c$ and $N \to \infty$. Simultaneous limits are standard knowledge in mathematics. In [3] we provided arguments that in order to obtain q-Gaussian limit distributions the simultaneous limit $a \to a_c$ and $N \to \infty$ must be performed in such a way that the scaling relation

$$N \sim 4^k \tag{3}$$

holds. Here k is again given by eq. (2), and $\delta = 4.6692011...$ is the Feigenbaum constant. In the notation used by Grassberger in [1] our scaling condition is equivalent to

$$N \sim n^2, \tag{4}$$

where again n denotes the number of chaotic bands. The abstract of [1] claims numerical inconsistency but the paper is misleading since most of the simulations in [1] ignore the above scaling condition (3) or (4) but operate in a different parameter region (called 'peaked region' in [3]). For example, for his results presented in Fig. 2 of [1] the author has chosen the distance $|a - a_c|$ from the critical point a_c to be of the order 10^{-18}, basically fixed by his numerical precision. For this value our scaling relation gives $k = 26.9$ and hence $N \sim 4^k \sim 2^{54} \sim 1.5 \cdot 10^{16}$ is required to see a q-Gaussian. On the contrary, the author performed his simulation in his Fig. 2 with the iteration numbers $N = 256, 2048, 16384$ and 131072, which are clearly insufficient to exhibit a q-Gaussian. Being that close to the critical point, much higher values of iteration numbers N are needed to obtain sufficient statistics to properly confirm or disconfirm our results presented in [3]. This is precisely the reason why in [3] we chose larger distances from a_c for which the relevant N-values are still reachable in a numerical experiment.

In Fig. 4 and 5 of [1] the author investigates band splitting points and mainly looks at cases in the parameter space given by $n >> \sqrt{N}$ or $n << \sqrt{N}$, again ignoring our scaling condition $n \sim \sqrt{N}$. Singular behavior of the density is observed simply because the scaling condition (4) is violated. Our present Fig. 1 shows how Fig. 4 of [1] would have looked like had the scaling condition been satisfied. In this case one gets smooth curves that are well-approximated by q-Gaussians. In this sense, for example the band splitting point $4096 \to 2048$ investigated in [1], which is the closest to the critical point, would yield the same q-Gaussian behavior had the iteration number $N = 2^{24}$ been used.

Let us provide a few further comments. In [1] some additional numerical experiments were performed testing the effects of different lengths N_0 of omitted transients. The author emphasizes that in his opinion the condition $N_0 >> N$ is relevant. We have checked this claim in the relevant parameter region fixed by $N \sim n^2$. As an example we have chosen one of the cases given in Fig. 2 of [3] (namely, $a = 1.401175$ and $N = 16384$) and tested the effect of discarding transients of length $N_0 = 2048, 4096, 8192, 16384, 65536$. The result is shown in Fig. 2. Apparently all curves fall onto each other, no matter whether $N_0 < N$ or $N_0 > N$, and are well approximated by a q-Gaussian. Hence the condition $N_0 >> N$ advocated in [1] seems to be irrelevant in the scaling region, as long as N_0 is sufficiently large (in [2, 3] N_0 was typically chosen to be 4096).

In [1] it is also stated that Lévy distributions, possibly motivated by the Lévy-Gnedenko limit theorem, could give equally good fits. To test this claim, we tried to fit our data by Lévy distributions as well. The result is shown in Fig. 3. The numerical data are well approximated by q-Gaussians, whereas Lévy distributions give worse fits. More specifically, in a log-log plot, Lévy distributions with $1 < \alpha < 2$ have an inflexion point which is by no means supported by the logistic-map data. If the parameter α of the Lévy distribution is slightly increased, then the fit quality in the middle region is slightly improved but the tails become too pronounced to provide an acceptable fit. Hence the claim of [1] that Lévy distributions might yield a better fit is incorrect. Besides this, the Lévy-Gnedenko limit theorem holds for *independent* (or nearly so) random variables with infinite variance, whereas the iterates of the logistic map near to the critical point have strong correlations. Hence there is no theoretical justification to use Lévy distributions in this problem. The conjecture in [1] that there might be a suitable ordering of the iterates into subsets that are almost independent lacks any theoretical proof or numerical justification.

Finally, in Fig. 1 of [1], new types of distributions of Y are shown for the case $N_0 = 0$, i.e. no transients are omitted, and claims are made at the end of the paper that these distributions including all the transients could be universal. As an argument for universality of transients at $a = a_c$, in [1] the work [4] is cited. In [4], however, only transient
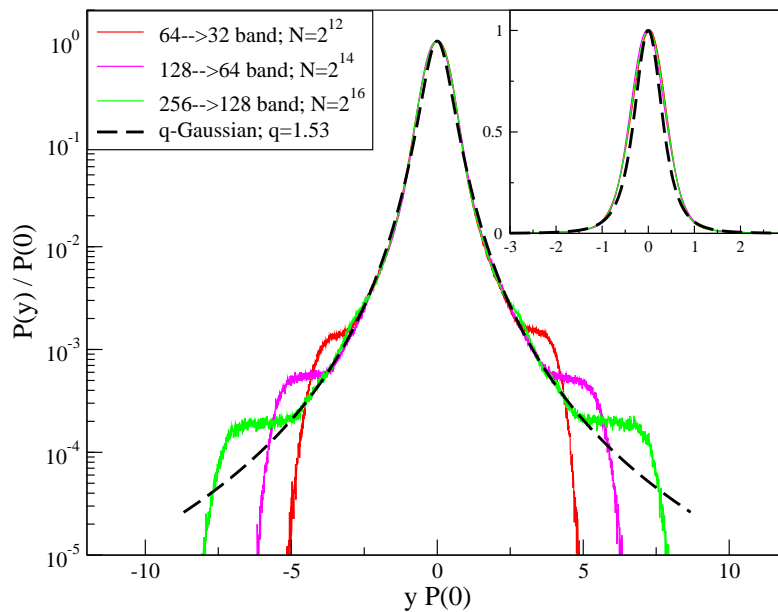
FIG. 1: Investigation of the density of Y for various band splitting points with number of iterations N satisfying the scaling condition $N \sim n^2$.
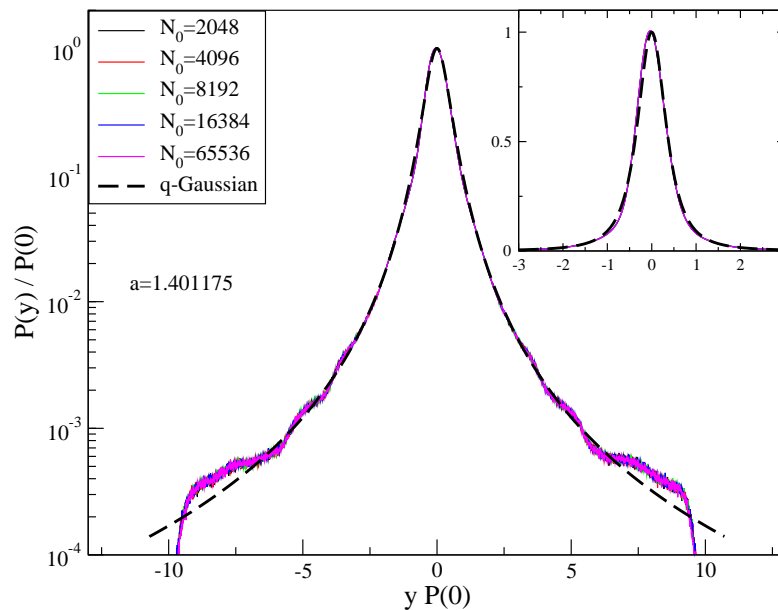

FIG. 2: Densities of Y obtained for various lengths N_0 of omitted transients in the scaling region $N \sim n^2$.

behaviour of iterates of the exact Feigenbaum fixed point function g is investigated, i.e. the map under consideration in [4] is the exact solution g of the Feigenbaum-Cvitanovic equation $\alpha g(g(x/\alpha)) = g(x)$. However, universal behavior in our case would mean that different maps f with quadratic maximum would generate the same distribution of Y. Since different quadratic maps can have very different transient behavior, it seems highly unlikely that the sum Y of all these different transients would converge to a random variable that has a universal distribution, as claimed in [1]. For this, one would have to carefully estimate the speed of convergence of the iterates of f to the Feigenbaum fixed point function g under successive iteration and rescalation, which was not done in [1]. Hence there is no theoretical basis for the claim of [1] that the observed *transient* distributions of sums are universal. Neither any numerical evidence of universality is provided in [1].

Summarizing, the numerical experiments of [1] were mainly performed in a different parameter region that was explicitly excluded by our scaling relation derived in [3]. In the parameter region chosen by Grassberger the number
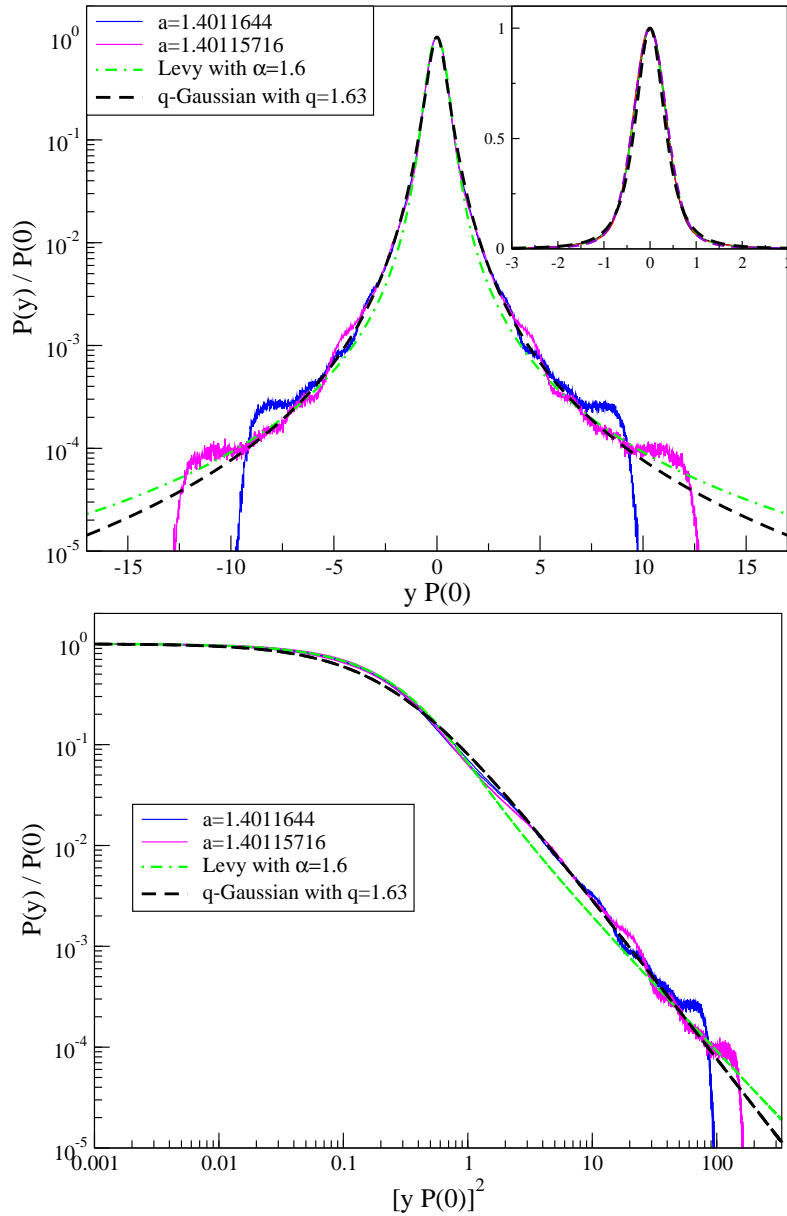
FIG. 3: Comparison of best fits obtained by using q-Gaussians ($q = 1.63$) and Lévy distributions ($\alpha = 1.60$), both in log-linear (a) and log-log plots (b).

N of iterations is insufficient. In the region fixed by our scaling argument $N \sim n^2$, q-Gaussians indeed provide good fits of the data, far better than the Lévy distributions suggested in [1], which moreover do not have any theoretical justification for this problem involving strongly correlated random variables. Transient distributions investigated in [1] are unlikely to be universal.

[1] P. Grassberger, Phys. Rev. E **79**, 057201 (2009).
[2] U. Tirnakli, C. Beck and C. Tsallis, Phys. Rev. E **75**, 040106 (2007).
[3] U. Tirnakli, C. Tsallis and C. Beck, Phys. Rev. E **79**, 056209 (2009).
[4] P. Grassberger and M. Scheunert, J. Stat. Phys. **26**, 697 (1981)