Queen Mary
UNIVERSITY OF LONDON

B.Sc. EXAMINATION

MAS/308 Chaos and Fractals

28 May 2002 14:30

Duration: 2 hours.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 3 questions will be counted.

Electronic calculators are not permitted.

1.

(a) Define generally what is meant by (i) a continuous-time dynamical system and (ii) a discrete-time dynamical system.

(b) Determine the trajectory $(x(t), y(t))$ of the dynamical system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$$

for a given initial condition $(x(0), y(0)) = (x_0, y_0)$.

(c) What is meant by saying that $f : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism? How many periodic orbits of length 1,2,7, respectively, can a 1-dimensional order-reversing diffeomorphism have?

(d) What does Sarkovskii's Theorem say?

(e) Suppose you have proved that some continuous 1-dimensional map has a periodic orbit of length L_1. Does this imply that the map also has a periodic orbit of length L_2? Consider the cases below.

i) $L_1 = 6$, $L_2 = 7$

ii) $L_1 = 2048$, $L_2 = 10240$

iii) $L_1 = 31234567$, $L_1 = 1024999993$.

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2.

(a) Define a fixed point of a 1-dimensional differentiable map f. Under which conditions is it stable?

(b) Same as (a), but for a d-dimensional map $\vec{f}$.

(c) Determine the fixed point of the logistic map as a function of μ.

(d) Determine the interval of μ where it is stable.

(e) The Henon map is given by

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n. \end{aligned}$$

Determine all fixed points of the Henon map. Write down the Jacobian. Determine the eigenvalues. Write down the conditions that must be satisfied in order to have a stable fixed point.

3.

(a) Define the Liapunov exponent of a 1-dimensional differentiable mapping f. Define 'chaos' for such a map.

(b) Define Liapunov exponents for d-dimensional differentiable maps.

(c) Determine the Liapunov exponents of Arnold's cat map

$$\begin{aligned} x_{n+1} &= (x_n + y_n) \bmod 1 \\ y_{n+1} &= (x_n + 2y_n) \bmod 1 \end{aligned}$$

Is the map chaotic?

(d) Define a partition of the phase space and explain the symbolic dynamics technique.

(e) Consider a mapping $f : X \to X$ and another mapping $g : Y \to Y$. What does it mean to say that f is topologically conjugated to g by means of a conjugacy h? Under which condition on h are Liapunov exponents of 1-dimensional maps invariant under topological conjugation?

4.

(a) What is an invariant measure of a chaotic map f? What is the relation between the invariant measure and the invariant density?

(b) Write down the definition of the Perron-Frobenius operator for a 1-dimensional differentiable map f. What equation is satisfied by a fixed point of the Perron-Frobenius operator? What is this fixed point for the binary shift map?

(c) Write down the Perron-Frobenius operator for the map $f(x) = 1 - 2\sqrt{|x|}$ on $X = [-1, 1]$ and show that the function

$$\rho(y) = C(1 - y)$$

is a fixed point of the Perron-Frobenius operator. Determine C by normalization.

(d) State the general definition of ergodicity.

(e) Calculate time averages

$$\bar{Q} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} Q(x_n)$$

of the observable $Q(x) = (1 - x)^2$ for

i) the binary shift map $f(x) = 2x \bmod 1$ on $X = [0, 1]$.

ii) the map $f(x) = 1 - 2\sqrt{|x|}$ on $X = [-1, 1]$.

You can assume that the maps are ergodic.

5.

(a) Define the fractal dimension (box dimension, capacity) of a fractal object.

(b) Describe the construction method of the Koch curve and of the Sierpinski gasket. Determine the corresponding fractal dimensions.

(Question continues on next page)

(c) Define the Mandelbrot set M of the complex logistic map.

(d) Define the Julia set J of the complex logistic map. If the Julia set is connected, what can you say about the corresponding parameter c? What is the Julia set for $c = 0$? How does it deform for small $|c| > 0$?

(e) Briefly describe a numerical method to determine the Julia set of the complex logistic map.