

Superstatistics: Recent developments and applications

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Abstract

We review some recent developments which make use of the concept of ‘superstatistics’, an effective description for nonequilibrium systems with a varying intensive parameter such as the inverse temperature. We describe how the asymptotic decay of stationary probability densities can be determined using a variational principle, and present some new results on the typical behaviour of correlation functions in dynamical superstatistical models. We briefly describe some recent applications of the superstatistics concept in hydrodynamics, astrophysics, and finance.

1 Introduction

Complex nonequilibrium systems often exhibit dynamical behaviour that is characterized by spatio-temporal fluctuations of an intensive parameter β . This intensive parameter may be the inverse temperature, or an effective friction constant, or the amplitude of Gaussian white noise, or the energy dissipation in turbulent flows, or simply a local variance parameter extracted from a signal. A nonhomogeneous spatially extended system with fluctuations in β can be thought of as consisting of a partition of spatial cells with a given β in each cell. If there is local equilibrium in each cell (so that statistical mechanics can be applied locally), and if the fluctuations of β evolve on a sufficiently large time scale, then in the long-term run the entire system is described by a superposition of different Boltzmann factors with different β , or in short, a ‘superstatistics’ [1]. The superstatistics approach has been the subject of various recent papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Superstatistical techniques can be successfully applied to a variety of physical problems, such as Lagrangian[14, 15] and Eulerian turbulence[16, 17], defect turbulence[18], cosmic ray statistics[19], plasmas[20], statistics of wind velocity differences[21, 22] and mathematical finance[23, 24, 25]. Experimentally measured non-Gaussian stationary distributions with ‘fat tails’ can often be successfully described by simple models that exhibit a superstatistical spatio-temporal dynamics.

If the intensive parameter in the various cells is distributed according to a particular probability distribution, the χ^2 -distribution, then the corresponding superstatistics, obtained by integrating over all β , is given by Tsallis statistics [26, 27, 28, 29]. For other distributions of the intensive parameter β , one ends up with more general superstatistics, which contain Tsallis statistics as a special case. Generalized entropies (analogues of the Tsallis entropies) can also be defined for general superstatistics[9, 12, 13, 30] and are indeed a useful tool. The ultimate goal is to proceed from simple models to more general versions of statistical mechanics, which are applicable to wide classes of complex nonequilibrium systems, thus further generalizing Tsallis’ original ideas[26].

This paper is organized as follows:

First, we briefly review the superstatistics concept. We then show how one can deduce the asymptotic decay rate of the stationary probability densities of general superstatistics from a variational principle[31]. In section 4 we consider dynamical realization of superstatistics and investigate the typ-

ical behaviour of correlation functions. Many different types of decays of correlations (e.g. power law, stretched exponentials) are possible. Section 5 summarizes some recent applications of the superstatistics concept in hydrodynamics, astrophysics, and finance.

2 What is superstatistics?

The superstatistics approach is applicable to a large variety of driven nonequilibrium systems with spatio-temporal fluctuations of an intensive parameter β , for example, the inverse temperature. Locally, i.e. in spatial regions (cells) where β is approximately constant, the system is described by ordinary statistical mechanics, i.e. ordinary Boltzmann factors $e^{-\beta E}$, where E is an effective energy in each cell. In the long-term run, the system is described by a spatio-temporal average of various Boltzmann factors with different β . One may define an effective Boltzmann factor $B(E)$ as

$$B(E) = \int_0^\infty f(\beta) e^{-\beta E} d\beta = \langle e^{-\beta E} \rangle, \quad (1)$$

where $f(\beta)$ is the probability distribution of β in the various cells. For so-called type-A superstatistics[1], one normalizes this effective Boltzmann factor and obtains the stationary long-term probability distribution

$$p(E) = \frac{1}{Z} B(E), \quad (2)$$

where

$$Z = \int_0^\infty B(E) dE. \quad (3)$$

For type-B superstatistics, the β -dependent normalization constant of each local Boltzmann factor is included into the averaging process. In this case the invariant long-term distribution is given by

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} e^{-\beta E} d\beta, \quad (4)$$

where $Z(\beta)$ is the normalization constant of $e^{-\beta E}$ for a given β . Eq. (4) is just a simple consequence of calculating marginal distributions. Type-B superstatistics can easily be mapped into type-A superstatistics by redefining $f(\beta)$.

A superstatistics can be dynamically realized by considering Langevin equations whose parameters fluctuate on a relatively large time scale (see [32] for details). For example, for turbulence applications one may consider a superstatistical extension of the Sawford model of Lagrangian turbulence [14, 15, 33]. This model consists of suitable stochastic differential equations for the position, velocity and acceleration of a Lagrangian test particle in the turbulent flow, and the parameters of this model then become random variables as well. Experimental data are well reproduced by these types of models.

Often, a superstatistics just consists of a superposition of Gaussian distributions with varying variance. The parameter β can then be estimated from an experimentally measured signal $u(t)$ as

$$\beta = \frac{1}{\langle u^2 \rangle_T - \langle u \rangle_T^2}, \quad (5)$$

where $\langle \dots \rangle_T$ denotes an average over a finite time interval T of the signal, corresponding to the ‘cell size’ of the superstatistics. It is then easy to make histograms of β and thus empirically determine $f(\beta)$.

3 Asymptotic behaviour for large energies

Superstatistical invariant densities, as given by eq.(1) or (4), typically exhibit ‘fat tails’ for large E , but what is the precise functional form of this large energy behaviour? The answer depends on the distribution $f(\beta)$ and can be obtained from a variational principle. Details are described in [31], here we just summarize some results. For large E we may use the saddle point approximation and write

$$\begin{aligned} B(E) &= \int_0^\infty f(\beta) e^{-\beta E} d\beta \\ &= \int_0^\infty e^{-\beta E + \ln f(\beta)} d\beta \\ &\sim e^{\sup_\beta \{-\beta E + \ln f(\beta)\}} \\ &= e^{-\beta_E E + \ln f(\beta_E)} \\ &= f(\beta_E) e^{-\beta_E E}, \end{aligned} \quad (6)$$

where

$$\beta_E = \sup_\beta \{-\beta E + \ln f(\beta)\}. \quad (7)$$

The expression

$$\sup_{\beta}\{-\beta E + \ln f(\beta)\} \quad (8)$$

corresponds to a Legendre transform of $\ln f(\beta)$. The result of this transform is a function of E which can be thought of as representing a kind of entropy function if we consider the function $\ln f(\beta)$ to represent a free energy function. This entropy function, however, is different from other entropy functions used e.g. in nonextensive statistical mechanics. It describes properties related to the fluctuations of inverse temperature.

In the case where $f(\beta)$ is smooth and has only a single maximum we can obtain the supremum by differentiating, i.e.

$$\sup_{\beta}\{-\beta E + \ln f(\beta)\} = -\beta_E E + \ln f(\beta_E) \quad (9)$$

where β_E satisfies the differential equation

$$0 = -E + (\ln f(\beta))' = -E + \frac{f'(\beta)}{f(\beta)}. \quad (10)$$

By taking into account the next-order contributions around the maximum, eq. (6) can be improved to

$$B(E) \sim \frac{f(\beta_E)e^{-\beta_E E}}{\sqrt{-(\ln f(\beta_E))''}}. \quad (11)$$

Let us consider a few examples. Consider an $f(\beta)$ of the power-law form $f(\beta) \sim \beta^\gamma$, $\gamma > 0$ for small β . An example is a χ^2 distribution of n degrees of freedom[32, 34],

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \quad (12)$$

($\beta_0 \geq 0$, $n > 1$) which behaves for $\beta \rightarrow 0$ as

$$f(\beta) \sim \beta^{n/2-1}, \quad (13)$$

i.e.

$$\gamma = \frac{n}{2} - 1. \quad (14)$$

Other examples exhibiting this power-law form are F -distributions[1, 20]. With the above formalism one obtains from eq. (10)

$$\beta_E = \frac{\gamma}{E} \quad (15)$$

and

$$B(E) \sim E^{-\gamma-1}. \quad (16)$$

These types of $f(\beta)$ form the basis for power-law generalized Boltzmann factors (q -exponentials) $B(E)$, with the relation [26, 27, 28, 29]

$$\gamma + 1 = \frac{1}{q-1}. \quad (17)$$

Another example would be an $f(\beta)$ which for small β behaves as $f(\beta) \sim e^{-c/\beta}$, $c > 0$. In this case one obtains

$$\beta_E = \sqrt{\frac{c}{E}} \quad (18)$$

and

$$B(E) \sim E^{-3/4} e^{-2\sqrt{cE}}. \quad (19)$$

The above example can be generalized to stretched exponentials: For $f(\beta)$ of the form $f(\beta) \sim e^{-c\beta^\delta}$ one obtains after a short calculation

$$\beta_E = \left(\frac{E}{c|\delta|} \right)^{1/(\delta-1)} \quad (20)$$

and

$$B(E) \sim E^{(2-\delta)/(2\delta-2)} e^{aE^{\delta/(\delta-1)}}, \quad (21)$$

where a is some factor depending on δ and c .

Of course which type of $f(\beta)$ is relevant depends on the physical system under consideration. For many problems in hydrodynamic turbulence, log-normal superstatistics seems to be working as a rather good approximation. In this case $f(\beta)$ is given by

$$f(\beta) = \frac{1}{\sqrt{2\pi s}\beta} \exp \left\{ -\frac{(\log \beta - m)^2}{2s^2} \right\}, \quad (22)$$

where s and m are parameters [1, 14, 15, 16, 17, 35].

4 Superstatistical correlation functions

To obtain statements on correlation functions, one has to postulate a concrete dynamics that generates the superstatistical distributions. The simplest dynamical model of this kind is a Langevin equation with parameters that vary on a long time scale, as introduced in [32].

Let us consider a Brownian particle of mass m and a Langevin equation of the form

$$\dot{v} = -\gamma v + \sigma L(t), \quad (23)$$

where v denotes the velocity of the particle, and $L(t)$ is normalized Gaussian white noise with the following expectations:

$$\langle L(t) \rangle = 0 \quad (24)$$

$$\langle L(t)L(t') \rangle = \delta(t - t'). \quad (25)$$

We assume that the parameters σ and γ are constant for a sufficiently long time scale T , and then change to new values, either by an explicit time dependence, or by a change of the environment through which the Brownian particle moves. Formal identification with local equilibrium states in the cells (ordinary statistical mechanics at temperature β^{-1}) yields during the time scale T the relation[36]

$$\langle v^2 \rangle = \frac{\sigma^2}{2\gamma} = \frac{1}{\beta m} \quad (26)$$

or

$$\beta = \frac{2}{m} \frac{\gamma}{\sigma^2}. \quad (27)$$

Again, we emphasize that after the time scale T , γ and σ will take on new values. During the time interval T , the probability density $P(v, t)$ obeys the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial(vP)}{\partial v} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial v^2} \quad (28)$$

with the local stationary solution

$$P(v|\beta) = \sqrt{\frac{m\beta}{2\pi}} \exp \left\{ -\frac{1}{2} \beta m v^2 \right\}. \quad (29)$$

In the adiabatic approximation, valid for large T , one assumes that the local equilibrium state is reached very fast so that relaxation processes can be

neglected. Within a cell in local equilibrium the correlation function is given by [36]

$$C(t - t'|\beta) = \langle v(t)v(t') \rangle = \frac{1}{m\beta} e^{-\gamma|t-t'|}. \quad (30)$$

Clearly, for $t = t'$ and setting $m = 1$ we have

$$\beta = \frac{1}{\langle v^2 \rangle_T}, \quad (31)$$

in agreement with eq. (5).

It is now interesting to see that the long-term invariant distribution $P(v)$, given by

$$P(v) = \int_0^\infty f(\beta)P(v|\beta)d\beta \quad (32)$$

depends only on the probability distribution of $\beta = \frac{2}{m} \frac{\gamma}{\sigma^2}$ and not on that of the single quantities γ and σ^2 . This means, one can obtain the same stationary distribution from different dynamical models based on a Langevin equation with fluctuating parameters. Either γ may fluctuate, and σ^2 is constant, or the other way round. On the other hand, the superstatistical correlation function

$$C(t - t') = \int_0^\infty f(\beta)C(t - t'|\beta)d\beta = \frac{1}{m} \int_0^\infty f(\beta)\beta^{-1}e^{-\gamma|t-t'|}d\beta \quad (33)$$

can distinguish between these two cases. The study of correlation functions thus yields more information for any superstatistical model.

Let illustrate this with a simple example. Assume that σ fluctuates and γ is constant such that $\beta = \frac{2}{m} \frac{\gamma}{\sigma^2}$ is χ^2 -distributed. Since γ is constant, we can get the exponential $e^{-\gamma|t-t'|}$ out of the integral in eq. (33), meaning that the superstatistical correlation function still decays in an exponential way:

$$C(t - t') \sim e^{-\gamma|t-t'|}. \quad (34)$$

On the other hand, if σ is constant and γ fluctuates and β is still χ^2 -distributed with degree n , we get a completely different answer. In this case, in the adiabatic approximation, the integration over β yields a power-law decay of $C(t - t')$:

$$C(t - t') \sim |t - t'|^{-\eta}, \quad (35)$$

where

$$\eta = \frac{n}{2} - 1 \quad (36)$$

Note that this decay rate is different from the asymptotic power law decay rate of the invariant density $P(v)$, which, using (29) and (32), is given by $P(v) \sim v^{-2/(q-1)}$, with

$$\frac{1}{q-1} = \frac{n}{2} + \frac{1}{2}. \quad (37)$$

In general, we may generate many different types of correlation functions for general choices of $f(\beta)$. By letting both σ and γ fluctuate we can also construct intermediate cases between the exponential decay (34) and the power law decay (35), so that strictly speaking we only have the inequality

$$\eta \geq \frac{n}{2} - 1, \quad (38)$$

depending on the type of parameter fluctuations considered.

One may also proceed to the position

$$x(t) = \int_0^t v(t') dt' \quad (39)$$

of the test particle. One has

$$\langle x^2(t) \rangle = \int_0^t \int_0^t \langle v(t') v(t'') \rangle dt' dt''. \quad (40)$$

Thus asymptotic power-law velocity correlations with an exponent $\eta < 1$ are expected to imply asymptotically anomalous diffusion of the form

$$\langle x^2(t) \rangle \sim t^\alpha \quad (41)$$

with

$$\alpha = 2 - \eta. \quad (42)$$

This relation simply results from the two time integrations.

It is interesting to compare our model with other dynamical models generating Tsallis statistics. Plastino and Plastino[37] and Tsallis and Bukmann[38] study a generalized Fokker-Planck equation of the form

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x}(F(x)P(x, t)) + D\frac{\partial^2}{\partial x^2}P(x, t)^\nu \quad (43)$$

with a linear force $F(x) = k_1 - k_2x$ and $\nu \neq 1$. Basically this model means that the diffusion constant becomes dependent on the probability density. The probability densities generated by eq. (43) are q -exponentials with the exponent

$$q = 2 - \nu. \quad (44)$$

The model generates anomalous diffusion with $\alpha = 2/(3 - q)$. Assuming the validity of $\alpha = 2 - \hat{\eta}$, i.e. the generation of anomalous diffusion by slowly decaying velocity correlations with exponent $\hat{\eta}$, one obtains

$$\hat{\eta} = \frac{4 - 2q}{3 - q}. \quad (45)$$

On the other hand, for the χ^2 -superstatistical Langevin model one obtains by combining eq. (36) and (37) the different relation

$$\eta = \frac{5 - 3q}{2q - 2}. \quad (46)$$

Interesting enough, there is a distinguished q -value where both models yield the same answer:

$$q = 1.453 \Rightarrow \hat{\eta} = \eta = 0.707 \quad (47)$$

These values of q and η correspond to realistic, experimentally observed numbers, for example in defect turbulence[18].

So far we mainly studied correlation functions with power law behaviour. But in fact one can construct superstatistical Langevin models that exhibit more complicated types of asymptotic behaviour of the correlation functions. To see this we notice that the asymptotic analysis of section 3 applies to correlation functions as well, by formally defining

$$\tilde{E} := \frac{1}{2}\sigma^2 m |t - t'| \quad (48)$$

$$\tilde{f}(\beta) = \frac{1}{m\beta} f(\beta) \quad (49)$$

and writing

$$C(t - t') = \int_0^\infty \tilde{f}(\beta) e^{-\beta \tilde{E}} d\beta. \quad (50)$$

To obtain statements on the symtpotic decay rate of the superstatistical correlation function, we may just use the same techniques described in section

3 with the replacement $E \rightarrow \tilde{E}$ and $f \rightarrow \tilde{f}$. In this way one can construct models that have, for example, stretched exponential asymptotic decays of correlations etc. (see also [39]). Asymptotic means here that $|t - t'|$ is large as compared to the local equilibrium relaxation time scale, but still smaller than the superstatistical time scale T , such that the adiabatic approximation is valid.

5 Some Applications

We end this paper by briefly mentioning some recent applications of the superstatistics concept. Rizzo and Rapisarda[21, 22] study experimental data of wind velocities at Florence airport and find that χ^2 -superstatistics does a good job. Jung and Swinney[35] study velocity differences in a turbulent Taylor-Couette flow, which is well described by lognormal superstatistics. They also find a simple scaling relation between the superstatistical parameter β and the fluctuating energy dissipation ϵ . Paczuski et al.[40] study data of solar flares on various time scales and embed this into a superstatistical model based on χ^2 -superstatistics = Tsallis statistics. Human behaviour when sending off print jobs might also stand in connection to such a superstatistics[41]. Bodenschatz et al.[42] have detailed experimental data on the acceleration of a single test particle in a turbulent flow, which is well described by lognormal superstatistics, with a Reynolds number dependence as derived in a superstatistical Lagrangian turbulence model studied by Reynolds[15]. The statistics of cosmic rays is well described by χ^2 -superstatistics, with $n = 3$ due to the three spatial dimensions[19]. In mathematical finance superstatistical techniques are well known and come under the heading ‘volatility fluctuations’, see e.g.[23] for a nice introduction and [24, 25] for some more recent work. Possible applications also include granular media, which could be described by different types of superstatistics, depending on the boundary conditions[43]. The observed generalized Tsallis statistics of solar wind speed fluctuations[44] is a further candidate for a superstatistical model. Chavanis[30] points out analogies between superstatistics and the theory of violent relaxation for collisionless stellar systems. Most superstatistical models assume that the superstatistical time scale T is very large, so that a quasi-adiabatic approach is valid, but Luczka and Zaborek[45] have also studied a simple model of dichotomous fluctuations of β where everything can be calculated for finite time scales T as well.

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