The combinatorics of the leading root of Ramanujan’s (and related) functions

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Combinatorics, Algebra, and More
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Topic Outline

1. q-Airy (and related) Functions
2. Identities for the Leading Roots
3. Combinatorics
4. Outlook
Outline

1. $q$-Airy (and related) Functions
   - Ramanujan’s Function
   - Painlevé Airy Function
   - Partial Theta Function

2. Identities for the Leading Roots

3. Combinatorics

4. Outlook
Ramanujan’s Function

\[ A_q(x) = \sum_{n=0}^{\infty} \frac{q^n (-x)^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q^n (-x)^n}{(1 - q)(1 - q^2) \ldots (1 - q^n)} \]
Ramanujan’s Function

\[ A_q(x) = \sum_{n=0}^{\infty} \frac{q^{n^2} \(-x)^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2} \(-x)^n}{(1 - q)(1 - q^2) \ldots (1 - q^n)} \]

Rogers-Ramanujan Identities

\[ A_q(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})} \]

and

\[ A_q(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})} \]

Related to partitions of integers into parts mod 5
Area-weighed Dyck paths

Count Dyck paths with respect to steps and enclosed area

\[ G(x, q) = \sum_{m, n} c_m, n x^n q^m = A_q(x) A_q(x/q) \]

- Counts pairs of up/down steps,
- Counts enclosed area
Count Dyck paths with respect to steps and enclosed area

Generating function

\[ G(x, q) = \sum_{m,n} c_{m,n} x^n q^m = \frac{A_q(x)}{A_q(x/q)} \]

\( x \) counts pairs of up/down steps, \( q \) counts enclosed area
\[ \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left( 1 - \frac{x q^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \ldots} \right) \]

where

\[ (t; q)_n = (1 - t)(1 - tq)(1 - tq^2)\ldots(1 - tq^{n-1}) \]

with \( y_1, y_2, y_3, y_4 \) explicitly given
\[
\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left( 1 - \frac{x q^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \ldots} \right)
\]

where

\[(t; q)_n = (1 - t)(1 - tq)(1 - tq^2) \ldots (1 - tq^{n-1})\]

with \(y_1, y_2, y_3, y_4\) explicitly given

More precisely,

\[y_1 = \frac{1}{(1 - q)\psi^2(q)}, \quad y_2 = 0\]

\[y_3 = \frac{q + q^3}{(1 - q)(1 - q^2)(1 - q^3)\psi^2(q)} - \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1 - q^{2n+1}} \frac{1}{(1 - q)^3\psi^6(q)}, \quad y_4 = y_1 y_3\]

and

\[\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}\]
The roots of $A_q(x)$

\[ \sum_{n=0}^{\infty} \frac{q^n (-x)^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left( 1 - \frac{xq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \ldots} \right) = \prod_{n=0}^{\infty} \left( 1 - \frac{x}{x_n(q)} \right) \]
The roots of $A_q(x)$

\[
\sum_{n=0}^{\infty} \frac{q^n (-x)^n}{(q; q)_n} = \prod_{n=1}^{\infty} \left( 1 - \frac{xq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \ldots} \right) = \prod_{n=0}^{\infty} \left( 1 - \frac{x}{x_n(q)} \right)
\]

- Roots are positive, real, and simple (Al-Salam and Ismail, 1983)
- Ramanujan’s expansion is an asymptotic series (Andrews, 2005)
- Relation to Stieltjes-Wigert polynomials (Andrews, 2005)
- Integral equation for roots (Ismail and Zhang, 2007)
- Combinatorial interpretation of $y_k$ (Huber, 2008, and Huber and Yee, 2010)
The aim of this talk

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q; q)_n} = \prod_{n=0}^{\infty} \left(1 - \frac{x}{x_n(q)}\right)
\]

\[
qx_0(-q) = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots
\]
The aim of this talk

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q; q)_n} = \prod_{n=0}^{\infty} \left(1 - \frac{x}{x_n(q)}\right)
\]

\[qx_0(-q) = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots\]

Goal

A combinatorial interpretation of the coefficients of the leading root
Painlevé Airy Function

\[
\text{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{q^n(-x)^n}{(1 - q^2)(1 - q^4)\ldots(1 - q^{2n})}
\]
Painlevé Airy Function

\[ \text{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(1 - q^2)(1 - q^4) \ldots (1 - q^{2n})} \]

\[ x_0(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 6q^5 + 12q^6 + 25q^7 + 54q^8 + 120q^9 + \ldots \]

The coefficients of the leading root also seem to be positive integers.
Painlevé Airy Function

\[ \text{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-x)^n}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-x)^n}{(1 - q^2)(1 - q^4) \ldots (1 - q^{2n})} \]
Painlevé Airy Function

\[
\text{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{(n)}_2(-x)^n}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{(n)}_2(-x)^n}{(1 - q^2)(1 - q^4) \ldots (1 - q^{2n})}
\]

Connection Formula (Morita, 2011)

\[
A_{q^2} \left( -\frac{q^3}{x^2} \right) = \frac{1}{(q; q)_{\infty}(-1; q)_{\infty}} \left\{ \Theta \left( -\frac{x}{q}, q \right) \text{Ai}_q(-x) + \Theta \left( \frac{x}{q}, q \right) \text{Ai}_q(x) \right\}
\]

with Theta Function

\[
\Theta(x, q) = \sum_{n=-\infty}^{\infty} q^{(n)}_2(-x)^n
\]
The Theta Function has roots $x_k(q) = q^k \quad k \in \mathbb{Z}$

$$\Theta(x, q) = \sum_{n=-\infty}^{\infty} q^{n^2/2} (-x)^n = (q; q)_\infty (x; q)_\infty (q/x; q)_\infty$$
The Theta Function has roots $x_k(q) = q^k \quad k \in \mathbb{Z}$

$$\Theta(x, q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} (-x)^n = (q; q)_\infty (x; q)_\infty (q/x; q)_\infty$$

The Partial Theta Function

$$\Theta_0(x, q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-x)^n$$

does not admit a “nice” product formula, but

$$x_0(q) = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots$$

The coefficients of the leading root are positive integers (Sokal, 2012)
Outline

1. \textit{q}-Airy (and related) Functions

2. Identities for the Leading Roots
   - The Leading Roots
   - Key Identities
   - Positivity

3. Combinatorics

4. Outlook
The Leading Roots

Partial Theta Function \( \Theta_0(x, q) = \sum_{n=0}^{\infty} q^{n/2} (-x)^n \)

\( \Theta_0(x, q) = 0 \)

\[ x = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots \]
The Leading Roots

Partial Theta Function $\Theta_0(x, q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-x)^n$

$\Theta_0(x, q) = 0$

$x = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots$

Ramanujan Function $A_q(x) = \sum_{n=0}^{\infty} q^n x^n/(q; q)_n$

$A_{-q}(x/q) = 0$

$x = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots$
### The Leading Roots

#### Partial Theta Function

\[
\Theta_0(x, q) = \sum_{n=0}^{\infty} q^{\frac{n}{2}} (-x)^n
\]

\[
\Theta_0(x, q) = 0
\]

\[
x = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots
\]

#### Ramanujan Function

\[
A_q(x) = \sum_{n=0}^{\infty} q^{n^2} (-x)^n / (q; q)_n
\]

\[
A_{-q}(x/q) = 0
\]

\[
x = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots
\]

#### Painlevé Airy Function

\[
Ai_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-x)^n / (q^2; q^2)_n
\]

\[
Ai_q(x) = 0
\]

\[
x = 1 + q + q^2 + 2q^3 + 3q^4 + 6q^5 + 12q^6 + 25q^7 + 54q^8 + 120q^9 + \ldots
\]
The Leading Roots

Key Identities

The combinatorics of the leading root of Ramanujan’s (and related) functions

\[ \Theta_0(x, q) = (x; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n x^n}{(q; q)_n (x; q)_n} \]
Key Identities

\( \Theta_0(x, q) \) satisfies (Andrews and Warnaar, 2007)

\[
\Theta_0(x, q) = (x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (x; q)_n}
\]

\( A_q(x) \) satisfies (Gessel and Stanton, 1983)

\[
A_{-q}(x/q) = (xq; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-n} x^n}{(q^2; q^2)_n (xq; q^2)_n}
\]
Key Identities

$\Theta_0(x, q)$ satisfies (Andrews and Warnaar, 2007)

$$
\Theta_0(x, q) = (x; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n(x; q)_n}
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$A_q(x)$ satisfies (Gessel and Stanton, 1983)

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A_{-q}(x/q) = (xq; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n} x^n}{(q^2; q^2)_n(xq; q^2)_n}
$$

$A_{i_q}(x)$ satisfies (Gessel and Stanton, 1983)

$$
A_{i_q}(x) = (x; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+\binom{n}{2}} x^n}{(q^2; q^2)_n(x; q)_n}
$$
## Identities for the Roots

**Partial Theta Function** (Sokal, 2012)

\[
\Theta_0(x, q) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^n x^n}{(q; q)_n (qx; q)_{n-1}}
\]
Partial Theta Function (Sokal, 2012)

\[ \Theta_0(x, q) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^n x^n}{(q; q)_n(qx; q)_{n-1}} \]

Ramanujan Function

\[ A_{-q}(x/q) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^n x^n}{(q^2; q^2)_n(q^2x; q^2)_{n-1}} \]
Identities for the Roots

Partial Theta Function

$$\Theta_0(x, q) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^n x^n}{(q; q)_n(qx; q)_{n-1}}$$

Ramanujan Function

$$A_{-q}(x/q) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^n x^n}{(q^2; q^2)_n(q^2x; q^2)_{n-1}}$$

Painlevé Airy Function

$$\text{Ai}_q(x) = 0 \text{ if } x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+\binom{n}{2}} x^n}{(q^2; q^2)_n(qx; q)_{n-1}}$$
Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q)_n (qx^{(N)}; q)_{n-1}}$$

Sokal (2012) shows coefficient-wise monotonicity of $x^{(N)}$, and hence positivity for the leading root of the Partial Theta Function.
Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q)_n (qx^{(N)}; q)_{n-1}}$$

Sokal (2012) shows coefficient-wise monotonicity of $x^{(N)}$, and hence positivity for the leading root of the Partial Theta Function

The proof is easily adapted to the two other identities
Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n(qx^{(N)}; q)_{n-1}}$$

Sokal (2012) shows coefficient-wise monotonicity of $x^{(N)}$, and hence positivity for the leading root of the Partial Theta Function.

The proof is easily adapted to the two other identities.

Is there an underlying combinatorial structure?
Outline

1. q-Airy (and related) Functions
2. Identities for the Leading Roots
3. Combinatorics
   - Ferrers Diagrams
   - Trees Decorated with Ferrers Diagrams
   - Changing the Area Weights
4. Outlook
a Ferrers diagram with Durfee square of size $n$
The generating function $G(x, y, q)$ of Ferrers diagrams with $n$-th largest row having length $n$ for some positive integer $n$, enumerated with respect to width ($x$), height ($y$), and total area ($q$), is given by

$$G(x, y, q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^n}{(qy; q)_n (qx; q)_{n-1}}$$
Why Ferrers Diagrams?

The generating function $G(x, y, q)$ of Ferrers diagrams with $n$-th largest row having length $n$ for some positive integer $n$, enumerated with respect to width $(x)$, height $(y)$, and total area $(q)$, is given by

$$G(x, y, q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^n}{(qy; q)_n (qx; q)_{n-1}}$$

Compare with

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (qx; q)_{n-1}}$$

to get

$$x = 1 + G(x, 1, q)$$
The combinatorics of the leading root of Ramanujan’s (and related) functions.

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The sum is over all sizes $n$ of Durfee squares.

$$G(x, y, q) = \sum_{n=1}^{\infty} (xy)^n q^{n^2} \frac{1}{(yq; q)_n} \frac{1}{(xq; q)_{n-1}}$$

must be width $n$
The functional equation

\[ x = 1 + G(x, 1, q) \]

admits a combinatorial interpretation using the “theory of species”

**Theorem (TP, 2012)**

Let \( F_q \) be the species of Ferrers diagrams with \( n \)-th largest row having length \( n \) for some integer \( n \), weighted by area \((q)\), with size given by the width of the Ferrers diagram, augmented by the ‘empty polyomino’. Then \( x \) enumerates \( F_q \)-enriched rooted trees (trees decorated such that the out-degree of the vertex matches the width of the Ferrers diagram) with respect to the total area of the Ferrers diagrams at the vertices of the tree.
Tree with area 15, contributing $q^{15}$ to the Partial Theta Function root
The Same Trees, But Different Area Weights

Partial Theta Function (TP, 2012)

\[
x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (q x; q)_{n-1}}
\]
The Same Trees, But Different Area Weights

Ramanujan Function (TP, 2013)

\[ x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}x^n}{(q^2; q^2)_n(q^2x; q^2)_{n-1}} \]

Count dark area twice
The Same Trees, But Different Area Weights

Painlevé Airy Function (TP, 2013)

\[ x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} + \binom{n}{2} x^n}{(q^2; q^2)_n (qx; q)_{n-1}} \]

Count dark area twice
Outline

1. *q*-Airy (and related) Functions
2. Identities for the Leading Roots
3. Combinatorics
4. Outlook
   - Why $\Theta, A_q, A_{i_q}$?
   - Higher Roots
   - Acknowledgement
Why are $\Theta$, $A_q$, $A_i q$ special?

Can we understand and generalise?
Why are $\Theta, A_q, A_i_q$ special?

Can we understand and generalise?

$$\sum_{n=0}^{N} \frac{(-1)^n q^{n+1}}{(q; q)_n(q; q)_{N-n}} = 1$$

follows from the $q$-binomial theorem.
Why are Θ, A_q, Ai_q special?

Can we understand and generalise?

\[ \sum_{n=0}^{N} \frac{(-1)^n q^{n+1}}{(q; q)_n (q; q)_{N-n}} = 1 \]

follows from the q-binomial theorem

\[ \sum_{n=0}^{N} \frac{q^n}{(q^2; q^2)_n (q^2; q^2)_{N-n}} = \frac{(-q, q)_N}{(q^2; q^2)_N} , \]

is given by Andrews (1976) and, more explicitly, by Cigler (1982)
Why are $\Theta, A_q, A_{iq}$ special?

Can we understand and generalise?

\[ \sum_{n=0}^{N} \frac{(-1)^n q^{n+1}}{(q; q)_n (q; q)_{N-n}} = 1 \]

follows from the $q$-binomial theorem

\[ \sum_{n=0}^{N} \frac{q^n}{(q^2; q^2)_n (q^2; q^2)_{N-n}} = \frac{(-q, q)_N}{(q^2; q^2)_N}, \]

is given by Andrews (1976) and, more explicitly, by Cigler (1982)

\[ \sum_{n=0}^{N} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n (q; q)_{N-n}} = \frac{1}{(q^2; q^2)_N}, \]

follows from an identity by Cauchy
Why are $\Theta$, $A_q$, $A_{iq}$ special?

Can we understand and generalise?

$$\sum_{n=0}^{N} \frac{(-1)^n q^{(n+1)/2}}{(q; q)_n (q; q)_{N-n}} = 1$$

follows from the $q$-binomial theorem

$$\sum_{n=0}^{N} \frac{q^n}{(q^2; q^2)_n (q^2; q^2)_{N-n}} = \frac{(-q, q)_N}{(q^2; q^2)_N},$$

is given by Andrews (1976) and, more explicitly, by Cigler (1982)

$$\sum_{n=0}^{N} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n (q; q)_{N-n}} = \frac{1}{(q^2; q^2)_N},$$

follows from an identity by Cauchy

These look similar, but their proofs have nothing in common!
Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q^2; q^2)_n(q^2 x; q^2)_{n-1}}$$
Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q^2; q^2)_n(q^2 x; q^2)_{n-1}}$$

Then

$$A_{-q}(y/q^{2m+1}) = 0 \text{ if } y = 1 + \frac{(-1)^m q^m}{\sum_{n=0}^{m} \frac{(-1)^n y^n q^n}{(q^2; q^2)_n} \prod_{k=1}^{m-n} (y - q^{2k})} \sum_{n=1}^{\infty} \frac{q^{n^2} y^{m+n}}{(q^2; q^2)_{m+n}(q^2 y; q^2)_{n-1}}$$
Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q^2; q^2)_n(q^2x; q^2)_{n-1}}$$

Then

$$A_{-q}(y/q^{2m+1}) = 0 \text{ if }$$

$$y = 1 + \sum_{n=0}^{m} \frac{(-1)^m q^m}{(q^2; q^2)_n} \sum_{k=1}^{m-n} \frac{(-1)^n y^n q^n}{(q^2q^2)^n} \prod_{k=1}^{m-n} (y - q^{2k}) \sum_{n=1}^{\infty} \frac{q^{n^2} y^{m+n}}{(q^2; q^2)_{m+n}(q^2y; q^2)_{n-1}}$$

Numerically, the $m$-th root seems to satisfy

$$\frac{1}{q^{2m+1}}(1 + (-1)^m q^{m+1}(\text{positive terms}))$$
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Thank you, Peter!