The combinatorics of the leading root of Ramanujan's (and related) functions

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Combinatorics, Algebra, and More
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Topic Outline

Outline

Ramanujan's Function

$$A_q(x) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(1-q)(1-q^2)\ldots(1-q^n)}$$

Ramanujan's Function

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Rogers-Ramanujan Identities

$$A_q(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

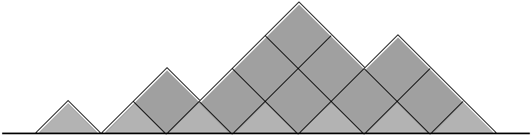
and

$$A_q(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

Related to partitions of integers into parts mod 5

Area-weighed Dyck paths

Count Dyck paths with respect to steps and enclosed area



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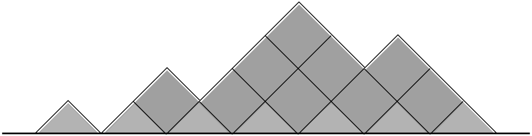

Generating function

$$G(x,q) = \sum_{m,n} c_{m,n} x^n q^m = \frac{A_q(x)}{A_q(x/q)}$$

x counts pairs of up/down steps, q counts enclosed area

Ramanujan's Lost Notebook, page 57

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 - \frac{x q^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \dots}\right)$$

where

$$(t;q)_n = (1-t)(1-tq)(1-tq^2)\dots(1-tq^{n-1})$$

with y_1, y_2, y_3, y_4 explicitly given

Ramanujan's Lost Notebook, page 57

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with y_1, y_2, y_3, y_4 explicitly given

More precisely,

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \quad y_2 = 0$$

$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \quad y_4 = y_1 y_3$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

The roots of $A_q(x)$

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$$= \prod_{n=0}^{\infty} \left(1 - \frac{x}{x_n(q)}\right)$$

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$$= \prod_{n=0}^{\infty}\left(1 - \frac{x}{x_n(q)}\right)$$

- Roots are positive, real, and simple (Al-Salam and Ismail, 1983)
- Ramanujan's expansion is an asymptotic series (Andrews, 2005)
- Relation to Stieltjes-Wigert polynomials (Andrews, 2005)
- Integral equation for roots (Ismail and Zhang, 2007)
- Combinatorial interpretation of y_k (Huber, 2008, and Huber and Yee, 2010)

The aim of this talk

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=0}^{\infty} \left(1 - \frac{x}{x_n(q)}\right)$$

$$qx_0(-q) = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots$$

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Goal

A combinatorial interpretation of the coefficients of the leading root

Painlevé Airy Function

$$\mathrm{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(1-q^2)(1-q^4)\dots(1-q^{2n})}$$

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$$x_0(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 6q^5 + 12q^6 + 25q^7 + 54q^8 + 120q^9 + \dots$$

The coefficients of the leading root also seem to be positive integers

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Connection Formula (Morita, 2011)

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q;q)_\infty(-1;q)_\infty}\left\{\Theta\left(-\frac{x}{q},q\right)\mathrm{Ai}_q(-x) + \Theta\left(\frac{x}{q},q\right)\mathrm{Ai}_q(x)\right\}$$

with Theta Function

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n$$

Partial Theta Function

The Theta Function has roots $x_k(q) = q^k \quad k \in \mathbb{Z}$

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n = (q;q)_\infty (x;q)_\infty (q/x;q)_\infty$$

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The Partial Theta Function

$$\Theta_0(x,q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$$

does not admit a "nice" product formula, but

$$x_0(q) = 1+q+2q^2+4q^3+9q^4+21q^5+52q^6+133q^7+351q^8+948q^9+\dots$$

The coefficients of the leading root are positive integers (Sokal, 2012)

Outline

1. q-Airy (and related) Functions
2. Identities for the Leading Roots
   - The Leading Roots
   - Key Identities
   - Positivity
3. Combinatorics
4. Outlook

The Leading Roots

Partial Theta Function $\Theta_0(x,q)=\sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$

$\Theta_0(x,q)=0$

$$x = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots$$

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Ramanujan Function $A_q(x)=\sum_{n=0}^{\infty} q^{n^2}(-x)^n/(q;q)_n$

$A_{-q}(x/q)=0$

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Key Identities

$\Theta_0(x,q)$ satisfies (Andrews and Warnaar, 2007)

$$\Theta_0(x,q) = (x;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n (x;q)_n}$$

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Identities for the Roots

Partial Theta Function (Sokal, 2012)

$\Theta_0(x,q)=0$ if

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n (qx;q)_{n-1}}$$

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Positivity

- Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n (qx^{(N)};q)_{n-1}}$$

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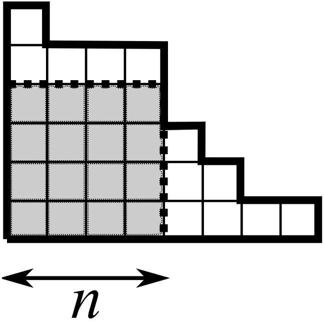
Is there an underlying combinatorial structure?

Outline

Ferrers Diagrams



a Ferrers diagram with Durfee square of size n

Why Ferrers Diagrams?

The generating function $G(x, y, q)$ of Ferrers diagrams with n-th largest row having length n for some positive integer n, enumerated with respect to width (x), height (y), and total area (q), is given by

$$G(x, y, q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^{n^2}}{(qy; q)_n (qx; q)_{n-1}}$$

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Compare with

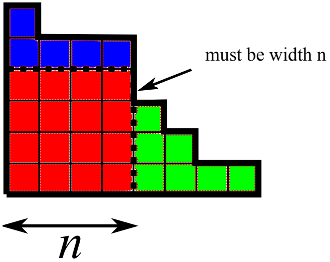
$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (qx; q)_{n-1}}$$

to get

$$x = 1 + G(x, 1, q)$$

Enumerating Ferrers Diagrams

$$G(x,y,q) = \sum_{n=1}^{\infty} (xy)^n q^{n^2} \frac{1}{(yq;q)_n} \frac{1}{(xq;q)_{n-1}}$$



The sum is over all sizes n of Durfee squares

Trees Decorated with Ferrers Diagrams

The functional equation

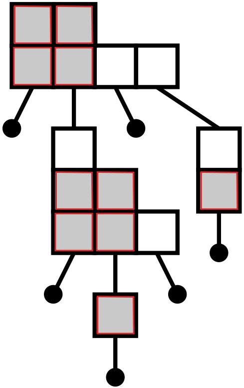
$$x = 1 + G(x, 1, q)$$

admits a combinatorial interpretation using the "theory of species"

Theorem (TP, 2012)

Let F_q be the species of Ferrers diagrams with n-th largest row having length n for some integer n, weighted by area (q), with size given by the width of the Ferrers diagram, augmented by the 'empty polyomino'. Then x enumerates F_q-enriched rooted trees (trees decorated such that the out-degree of the vertex matches the width of the Ferrers diagram) with respect to the total area of the Ferrers diagrams at the vertices of the tree.

Trees Decorated with Ferrers Diagrams



Tree with area 15, contributing q^{15} to the Partial Theta Function root

The Same Trees, But Different Area Weights

Partial Theta Function (TP, 2012)

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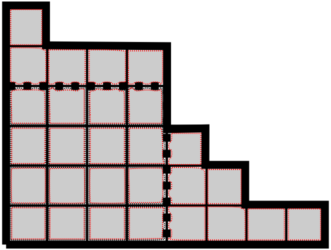

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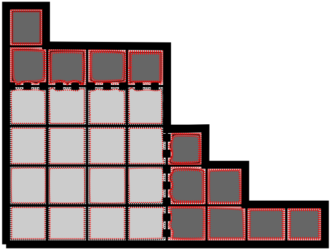

Count dark area twice

The Same Trees, But Different Area Weights

Painlevé Airy Function (TP, 2013)

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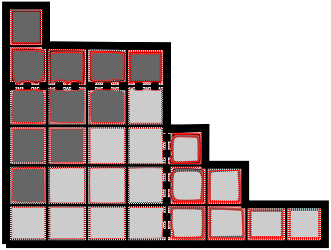

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Outline

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2. Identities for the Leading Roots
3. Combinatorics
4. Outlook
   - Why Θ, A_q, Ai_q?
   - Higher Roots
   - Acknowledgement

Why are Θ, A_q, Ai_q special?

Can we understand and generalise?

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$$\sum_{n=0}^{N} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n (q;q)_{N-n}} = 1$$

follows from the q-binomial theorem

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is given by Andrews (1976) and, more explicitly, by Cigler (1982)

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These look similar, but their proofs have nothing in common!

Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

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$A_{-q}(y/q^{2m+1}) = 0$ if

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Numerically, the m-th root seems to satisfy

$$\frac{1}{q^{2m+1}} (1 + (-1)^m q^{m+1} (\text{positive terms}))$$

Acknowledgement

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Thank you, Peter!