Optimal Resistor Networks

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Resistor Networks

Have a graph G (possibly with multiple edges).

Vertex set V, edge (multi-)set E. Usually $|V| = n$, $|E| = m$.

Form an electrical network by replacing each edge of G with a 1 ohm resistor.

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Form an electrical network by replacing each edge of G with a 1 ohm resistor.

We are interested in R_{xy}, the effective resistance between a pair of vertices $x, y \in V(G)$.

Physically, connect a 1 volt battery across x, y, measure current I flowing, $R_{xy} = I^{-1}$.

(Have series and parallel laws etc.)

Mathematical approach to R_{xy}

- Algebraic definition (Kirchoff's Laws)
  Want currents $(u_e)_{e \in E}$, $(u_{ab} = -u_{ba})$ with

$$\sum_{t \in \Gamma(a)} u_{at} = \begin{cases} 0 & \text{if } a \neq x, y \\ I & \text{if } a = x \\ -I & \text{if } a = y \end{cases}$$

  and potentials $(p_v)_{v \in V}$ with $u_{ab} = p_a - p_b$, $p_x = 1, p_y = 0$.
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- Graph theoretic formula

$$R_{xy} = \frac{\#\{\text{2 component spanning forests separating } x \text{ from } y\}}{\#\{\text{spanning trees}\}}$$

Some Properties

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  - Identifying 2 vertices can never increase R_{xy}.

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  Follow easily from analytic formulation.
- Triangle inequality

  For any $x, y, z \in V$

$$R_{xy} \leq R_{xz} + R_{zy}.$$

  Follows easily from the graph theoretic formulation.

An Application from Statistics

Let $v_1, \ldots, v_n$ be unknown quantities.
For each $e = (i, j) \in E(G)$, measure the quantity

$$v_i - v_j + N_e$$

where N_e are i.i.d. random variables with

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The best G to use will be one in which the R_{xy} are small given the number of edges is m.

Main Question

Let G be a graph with $|V| = n$.
Define
$$A(G) = \frac{1}{\binom{n}{2}} \sum_{\{x,y\}\subset V} R_{xy}.$$

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Question

Given G with $|V| = n, |E| = m$, how small can $A(G)$ be?

Notation:

$$a(n, m) = \min\{A(G) : |V(G)| = n, |E(G)| \leq m\}$$

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Because of this we will focus on the asymptotic behaviour.

$$a(\alpha) = \lim_{n \to \infty} a\left(n, \frac{\alpha n}{2}\right)$$

By the above $a(2) = 2$.

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$$a(\alpha) = \lim_{n\to\infty} a\left(n, \frac{\alpha n}{2}\right)$$

By the above $a(2) = 2$.
For $\alpha \in [2, 3]$ a star of triangles and leaves gives the bound:

$$a(\alpha) \leq 2 - \frac{2(\alpha - 2)}{3}; \quad a(3) \leq 4/3.$$

Larger m: Random Regular Constructions

For $\alpha \in \{3, 4, 5, \dots\}$, let G be a random α-regular graph on n vertices.
Suppose that G has girth $\geq \log\log n$ (happens with positive probability).
Heuristic: For most $x, y \in V$

$$R_{x,y} \sim 2\left(\frac{1}{\alpha} + \frac{1}{\alpha(\alpha-1)} + \frac{1}{\alpha(1-\alpha)^2} + \dots\right) + o(1) = \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1)$$

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This is not a proof.
But, a slightly ugly modification gives graphs with average degree α and $A(G) = \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1)$
For $\alpha \in \{3, 4, 5, \dots\}$ this gives the bound

$$a(\alpha) \leq \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1).$$

Lower Bounds

A lower bound of

$$a(\alpha) \geq \frac{2}{\alpha}.$$

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$$a(\alpha) \stackrel{?}{\geq} \frac{2(\alpha - 1)}{\alpha(\alpha - 2)} + o(1).$$

But this is false; there are graphs with $\alpha = 3$ and $A(G) < 4/3$.

Rooted Version

Now consider graphs with a special root vertex r.

Let

$$B(G) = \frac{1}{|V| - 1} \sum_{x \in V} R_{xr}.$$

$$b(n,m) = \min\{B(G) : |V(G)| = n+1, |E(G)| \leq m\}$$

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This rooted version is closely related to the unrooted version.

Theorem

- *For any G we have $A(G) \leq 2B(G)$.*
- *For any n vertex G_n there is an $n + 1$ vertex H_n with*

$$2B(H_n) \leq A(G_n) + o(1).$$

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Moreover, given 2 graphs

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by identifying their root vertices we get a new graph G_3 with

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Extending this, we get that $b(\alpha)$ is convex (and hence continuous) on $[2, \infty)$ and so:

Theorem

The function $a(\alpha)$ exists and is convex on $[2, \infty)$.

Note: we do not see how to prove this without using the rooted version.

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In fact graphs with $\alpha = 10/3$ and $B(G) \approx 0.527186$ exist and are sufficient to give better graphs for all $\alpha \in (2, 10/3]$.
In particular:

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In particular:

Theorem
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The $\alpha = 10/3$ graphs involved are random (large girth) graphs with all degrees 3 and 4 and some adjacency constraints.

Two Conjectures

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Conjecture

If G *is a random* α*-regular graph on* n *vertices then with high probability* $A(G) = \frac{2(\alpha-1)}{\alpha(\alpha-2)} + o(1)$.