Harmonic Ricci Flow on surfaces

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(joint work with Melanie Rupflin)

Let $g(t)$ be a family of smooth Riemannian metrics on an $n$-dimensional closed manifold $M$. Moreover, given a smooth closed Riemannian manifold $(N, g_N)$ of arbitrary dimension, let $\phi(t)$ be a family of smooth maps from $M$ to $N$. Then $(g(t), \phi(t))$ is called a solution of the volume preserving Harmonic Ricci Flow (or Ricci Flow coupled with Harmonic Map Heat Flow), if it satisfies

$$
\begin{array}{l}
\frac{\partial}{\partial t} g = -2 \text{Ric}_g + 2\alpha d\phi \otimes d\phi + \frac{2}{n} \int_M \left( R_g - \alpha |d\phi|_g^2 \right) d\mu_g =: T(g, \phi), \\
\frac{\partial}{\partial t} \phi = \tau_g(\phi).
\end{array}
$$

(0.1)

Here, Ric$_g$ and $R_g$ denote the Ricci and scalar curvatures of $(M, g)$, $\alpha$ is a (possibly time-dependent) positive coupling constant, and $\tau_g(\phi) = \text{tr}_g(\nabla d\phi)$ is the tension field of $\phi$.

The Harmonic Ricci Flow was introduced in [4], with some special cases previously studied in [2, 3]. Some of the key properties of this flow are that on the one hand, in special situations, it behaves less singular than the two flows considered separately, while on the other hand most of the Ricci Flow techniques carry over almost directly to the coupled system. Therefore, this relatively new flow has gained the attention of many authors recently, studying the flow usually in general dimensions. In this talk, we consider the special case where the domain manifold $M$ is a surface of positive genus $\gamma > 0$, a situation in which much stronger results can be obtained. In particular, we will explain that at a finite singular time of the flow, both the map and the metric component must blow up simultaneously.

**Theorem 0.1** (Theorem 1.2. of [1])

Let $M$ be a closed surface and let $(g, \phi)$ be a solution of (0.1) defined and smooth on a maximal time interval $[0, T)$ and with a smooth coupling function $\alpha$ that is bounded away from zero. If $T < \infty$, then

$$
\lim_{t \to T} \sup_{x \in M} |K_g(t)(x)| = \infty \quad \text{and} \quad \lim_{t \to T} \sup_{x \in M} \frac{1}{2} |d\phi(x, t)|^2_{g(t)} = \infty,
$$

where $K_g$ denotes the Gauss curvature of $(M, g)$.

If the coupling constant $\alpha$ is chosen large enough, such finite time singularities cannot happen. We prove the following theorem.

**Theorem 0.2** (Theorem 1.1. of [1])

Let $\alpha(t) \in [\alpha_0, \bar{\alpha}]$ be a smooth coupling function, where $0 < \alpha_0 \leq \bar{\alpha} < \infty$ and

$$
\alpha > 2 \max \left\{ K(\tau) \mid \tau \in T_pN \text{ two-plane, } p \in N \right\}.
$$

(0.2)
Here, $K(\tau)$ denotes the sectional curvature of the target manifold $(N, g_N)$ at a point $p$ in direction $\tau$. Then every solution $(g, \phi)$ of (0.1) with a two-dimensional domain manifold is defined and smooth for all times $t \geq 0$.

Theorem 0.2 follows directly from Theorem 0.1, as the assumption (0.2) prevents $|d\phi|^2_g$ from blowing up. In fact, by the Bochner-formula, $|d\phi|^2_g$ is uniformly bounded (in space and time) in terms of its initial value and $\bar{\alpha}, \bar{\alpha}$ satisfying (0.2). Moreover, in [4, Corollary 5.3], we showed that if the coupling constant $\alpha(t)$ is smooth and bounded away from zero, then a concentration of $|d\phi|^2_g$ cannot happen as long as the curvature of $g(t)$ stays bounded. Thus in order to prove Theorem 0.1, we need to show that in the case of a two-dimensional domain, the converse holds as well, that is, it is not possible for the (Gauss) curvature $K_{g(t)}$ of $g(t)$ to blow up while $|d\phi|^2_g$ remains bounded. In other words, both Theorems 0.1 and 0.2 are consequences of the following main result, which is equivalent to Proposition 1.3. in [1].

**Theorem 0.3** (Proposition 1.3. of [1])

Let $(g, \phi)$ be a solution of (0.1) and assume that on an interval $[0, T), T < \infty$, we have

$$\sup_{x \in M, t \in [0, T)} \frac{1}{2} |d\phi(x, t)|^2_{g(t)} < \infty. \quad (0.3)$$

Then both the curvature and the injectivity radius of $g(t)$ are uniformly bounded,

$$\sup_{x \in M, t \in [0, T)} |K_{g(t)}(x)| < \infty, \quad \text{and} \quad \inf_{t \in [0, T)} \text{inj}(M, g(t)) > 0,$$

and thus the solution $(g, \phi)$ of (0.1) can be extended smoothly past time $T$.

The main idea used to prove Theorem 0.3 is that one can always split a flow of metrics on a surface into a conformal part, the pull-back by diffeomorphisms and a horizontal movement. More precisely, there exist a family of smooth diffeomorphisms $f_t$ of $M$, a smooth function $u(t)$ and a horizontal curve $g_0(t)$, such that

$$g(t) = f_t^* \left( e^{2u(t)} g_0(t) \right). \quad (0.4)$$

We then first show that the evolution of the underlying conformal structure, described by the horizontal curve $g_0(t)$, is well controlled and in particular that the injectivity radius of $g_0(t)$ is a priori bounded away from zero on any given time interval of finite length by the theory of Rupflin and Topping on Teichmüller Harmonic Map Flow, see in particular [5, 6] and references therein.

Next, we analyse the evolution of the conformal factor $u(t)$ following the approach of Struwe [7], i.e. by studying the Liouville energy

$$E_L(t) := \frac{1}{2} \int_M \left( |du(t)|^2_{g_0(t)} + 2\bar{K} u(t) \right) d\mu_{g_0(t)}, \quad (0.5)$$

where $\bar{K}$ is the average Gauss curvature of $(M, g)$. The main differences to the Ricci Flow case studied in [7] are that the background metric $g_0$ is not fixed, but an evolving horizontal curve, and that the evolution equation for $u(t)$ contains various extra terms stemming from the map component of the flow (0.1) as well as the diffeomorphisms $f_t$. Nevertheless, we can still derive bounds on the Liouville energy in this more complicated situation and
they in turn then yield estimates on the $H^1$-norm of $u$. In a further step, we also derive $H^2$-bounds on $u$ with respect to the evolving background metric $g_0(t)$, before setting up a bootstrapping scheme to obtain higher regularity estimates and conclude in particular the claimed curvature and injectivity radius bounds for $g(t)$.

Once uniform bounds on the curvature, the injectivity radius and the energy density are known, a solution $(g,\phi)$ of (0.1) can always be smoothly extended by standard arguments – compare with Section 6 of [4] where the corresponding result was proven in detail for the non-renormalised Harmonic Ricci Flow in arbitrary dimension.

References


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