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MAS200 Actuarial Statistics

Whole-life annuities revisited Spring 2002

The present value of a life annuity is a random variable. It can be expressed via I_t, the indicator-of-survival of x to age $x+t$. For example, the present value of a whole-life annuity-due to (x) which pays 1 unit of money annually for life is $z = \sum_{k=0}^{\infty} v^k I_k$. This representation is helpful when one calculates the expectation of z: $E(\sum_{k=0}^{\infty} v^k I_k) = \sum_{k=0}^{\infty} v^k E(I_k)$ and $E(I_k) = {}_kp_x$, so that

$$E(z) = \sum_{k=0}^{\infty} v^k \, {}_kp_x \tag{1}$$

However, when one wants to calculate the variance of present value, the representation $z = \sum_{k=0}^{\infty} v^k I_k$ and similar representations for other types of annuitites do not help. (Notice that the random variables I_k, $k = 0, 1, \ldots$, are not independent, and hence $\mathrm{var}(z) = \mathrm{var}(\sum_k v^k I_k) \neq \sum_k v^{2k} \mathrm{var}(I_k)$.)

It is often more convenient to express the present values of life annuities directly in terms of exact time-until-death, $T(x)$, or its integer part, $K(x)$, known as the curtate time-until-death.

Recall from Lecture 5 (see handout "Cash flows") that the present value of a continuous flow of money,at the rate 1 p.a., over time interval $[0, t]$ is

$$\bar{a}_{\overline{t|}} = \int_0^t v^u \, du = \frac{1 - v^t}{\delta}, \qquad \delta = -\ln v = \ln(1+i) \tag{2}$$

If (x) is entitled to 1 p.a. for life payable continuously, the corresponding payment is a whole-life annuity. On the other hand, this payment is to be made continuously over the time interval $[0, T(x)]$, hence its present value is $\bar{a}_{\overline{T(x)|}}$.

Therefore:

the present value of a whole-life annuity payable continuously, at rate 1 p.a., until the moment of death of (x) is $\bar{a}_{\overline{T(x)|}}$, hence $\bar{a}_x = E(\bar{a}_{\overline{T(x)|}})$.

The present value of an annuity-due of 1 p.a. payable annually (at the beginning of each year) is

$$\ddot{a}_{\overline{k+1|}} = 1 + v + \ldots v^k = \frac{1 - v^{k+1}}{1 - v} = \frac{1 - v^{k+1}}{d}, \tag{3}$$

where $k+1$ is the number of payments and d is the effective rate of discount.

Therefore, similarly to annuities payable continuously:

$\ddot{a}_{\overline{K(x)+1|}}$ is the present value of a whole-life annuity of 1 p.a. payable annually in advance until the death of (x). As the annuity is payable in advance, the first installment is due at the present time (i.e. now) and is payable even if $K(x)=0$, hence $K(x)+1$, not $K(x)$. Obviously, $\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1|}})$.

Expected present value of annuities revisited:

Let us first re-derive the equation:

$$\bar{a}_x = \int_0^\infty v^T \ {}_tp_x \, dt \tag{4}$$

which we obtained in Lecture 23 by letting $p \to \infty$ in the expression for $\ddot{a}_x^{(p)}$.

Here, we shall obtain Eq. (4) directly from the definiton of $\bar{a}_x$:

$$\bar{a}_x = E(\bar{a}_{\overline{T(x)|}}) = \int_0^\infty \bar{a}_{\overline{t|}} \, f_{T(x)}(t)\, dt = \int_0^\infty \bar{a}_{\overline{t|}} \ {}_tp_x \mu(x+t) dt,$$

where we have used that $f_{T(x)}(t) = {}_tp_x\mu(x+t)$, $t \geq 0$. It can be verified by a straightforward calculation of derivatives that

$$\frac{d}{dt}\left({}_tp_x\right) = -{}_tp_x\mu(x+t).$$

Therefore ${}_tp_x\mu(x+t)dt = -d({}_tp_x)$, and by making use of integration by parts,

$$\bar{a}_x = -\int_0^\infty \bar{a}_{\overline{t|}}\, d({}_tp_x) = -\left[\bar{a}_{\overline{t|}}\ {}_tp_x\right]_{t=0}^\infty + \int_0^\infty {}_tp_x\left(\frac{d}{dt}\bar{a}_{\overline{t|}}\right)dt.$$

Now notice that $\left[\bar{a}_{\overline{t|}}\ {}_tp_x\right]_{t=0}^\infty = 0$ and $\frac{d}{dt}\bar{a}_{\overline{t|}} = v^t$ (follows from Eq. (2)). Therefore

$$\bar{a}_x = E(\bar{a}_{\overline{T(x)|}}) = \int_0^\infty v^t \ {}_tp_x \, dt,$$

the same expression as in Eq. (4).

Now whole-life annuities-due.

$\ddot{a}_{\overline{K(x)+1|}}$ is a discrete-type random variable that takes values $\ddot{a}_{\overline{k+1|}}$ with the probabilities $P(K(x)=k) = {}_kp_x - {}_{k+1}p_x$ (see Claim 1 in Lecture 13). Therefore

$$\begin{aligned}\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1|}}) &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} P(K(x)=k) \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} ({}_kp_x - {}_{k+1}p_x).\end{aligned}$$

Now notice that $\ddot{a}_{\overline{k+1|}} = \ddot{a}_{\overline{k|}} + v^k$ for all integer k (follows from Eq. (3)). Therefore

$$\begin{aligned}E(\ddot{a}_{\overline{K(x)+1|}}) &= \sum_{k=0}^{\infty} (\ddot{a}_{\overline{k|}} + v^k)({}_kp_x) - \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}}({}_{k+1}p_x) \\ &= 1 + \sum_{k=1}^{\infty} v^k\, {}_kp_x \qquad [\text{as } \ddot{a}_{\overline{0|}} = 0]\end{aligned}$$

Thus finally,

$$\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1|}}) = \sum_{k=0}^{\infty} v^k\, {}_kp_x,$$

the same expression as in Eq. (1).

Variance of the present value of annuities

We want to express $\mathrm{var}(\bar{a}_{\overline{T(x)|}})$, variance of the present value of a whole-life annuity payable continuously, and $\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}})$, variance of the present value of a whole-life annuity payable annually in advance, in terms of life table functions.

From Probability I, for any random variable X and two constants α and β: $\mathrm{var}(\alpha X + \beta) = \alpha^2 \mathrm{var}(X)$. We will use this fact and Eqs. (2) and (3) to find $\mathrm{var}(\bar{a}_{\overline{T(x)|}})$ and $\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}})$

By Eq. (2),

$$\mathrm{var}(\bar{a}_{\overline{T(x)|}}) = \frac{1}{\delta^2}\mathrm{var}(v^{T(x)}).$$

Recall from Lecture 19 that $v^{T(x)}$ is the present value of unit benefit payable immediately on the death of (x) under a whole-life assurance contract. The expectation of this present value is $\bar{A}_x$ and its variance is $\bar{A}_x^* - (\bar{A}_x)^2$, where $\bar{A}_x = E(v^{T(x)})$ and the star refers to $v^* = v^2$. Therefore

$$\mathrm{var}(\bar{a}_{\overline{T(x)|}}) = \frac{\bar{A}_x^* - (\bar{A}_x)^2}{\delta^2}. \tag{5}$$

Similarly, by Eq. (3),

$$\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}}) = \frac{1}{d^2}\mathrm{var}(v^{K(x)+1}),$$

where $v^{K(x)+1}$ is the present value of one unit of benefit payable at the end of the year of death under a whole-life assurance contract (see Lecture 21) and $\mathrm{var}(v^{K(x)+1}) = A_x^* - (A_x)^2$. Therefore,

$$\mathrm{var}(\ddot{a}_{\overline{K(x)+1}|}) = \frac{A_x^* - (A_x)^2}{d^2}. \tag{6}$$

Conversion relationships revisited:

$$\begin{aligned} \bar{A}_x &= 1 - \delta \bar{a}_x & (7) \\ A_x &= 1 - d\ddot{a}_x & (8) \end{aligned}$$

Eqs. (7) and (8) follow from Eqs. (2) and (3). Indeed, from Eq. (2), $v^{T(x)} = 1 - \delta \bar{a}_{\overline{T(x)}|}$ and from Eq. (3), $v^{K(x)+1} = 1 - d\ddot{a}_{\overline{K(x)+1}|}$. Taking the expectation one obtains Eqs. (7) and (8).