Non-Hermitian random matrices

Boris Khoruzhenko
School of Mathematical Sciences
Queen Mary, University of London
B.Khoruzhenko@qmw.ac.uk
http://www.maths.qmw.ac.uk/~boris

The Diablerets Winter School
"Random Matrices"
(18-23 March 2001)

Plan:

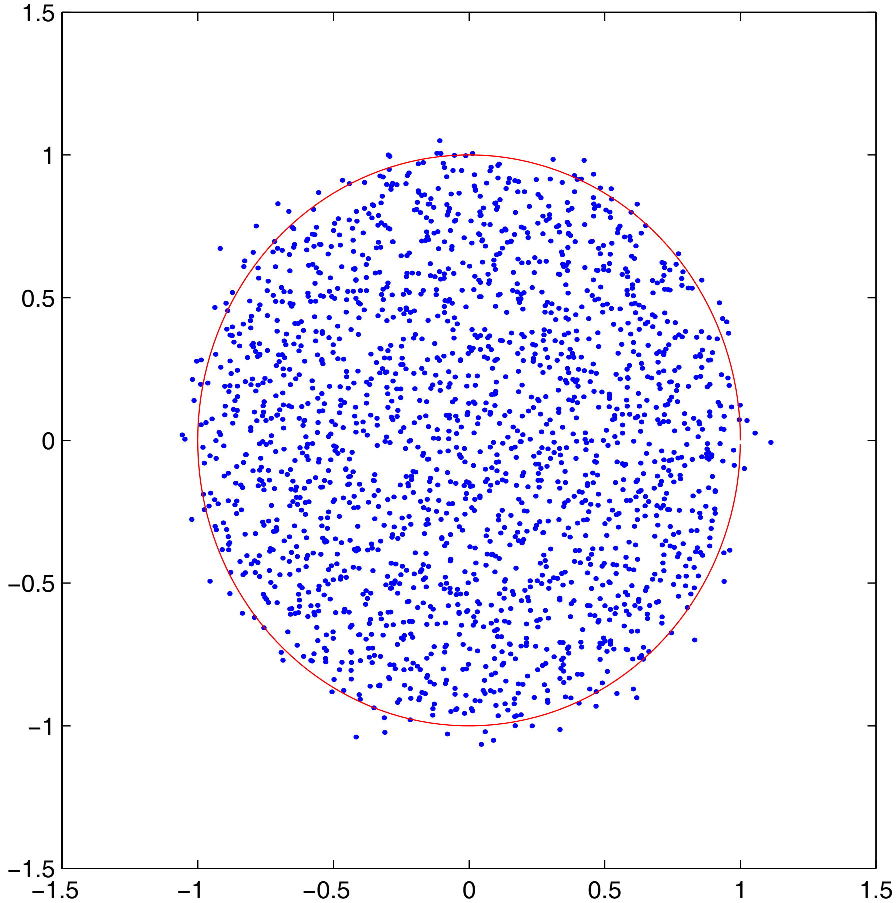
- Survey of non-Hermitan random matrices (ensembles, tools, results, open problems)
- Weakly non-Hermitian random matrices
- Asymmetric tridiagonal random matrices

Part I. Survey of non-Hermitian random matrices

only slides with results of numerical experiments are available at present

Circular distribution of eigenvalues

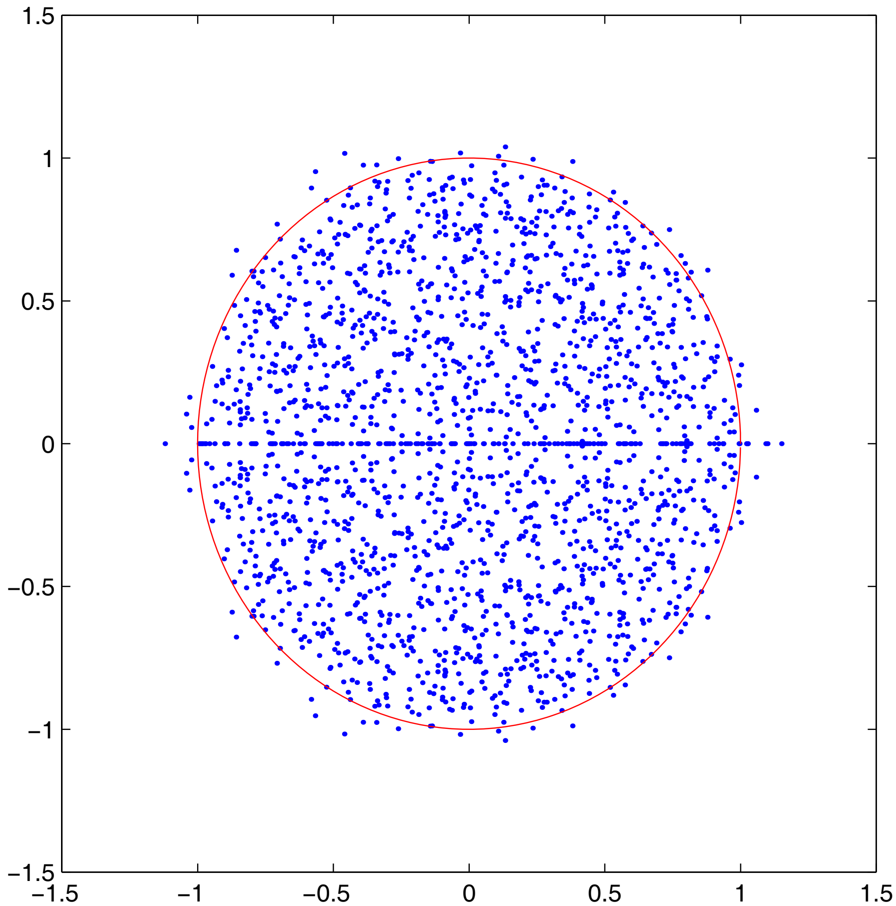
(complex matrices)



Normalized eigenvalues of 20 complex matrices of size $n = 100$ represented by dots. Matrix entries are "drawn" independently from $N(0, 1/2) + i\text{independent}N(0, 1/2)$.

Circular distribution of eigenvalues

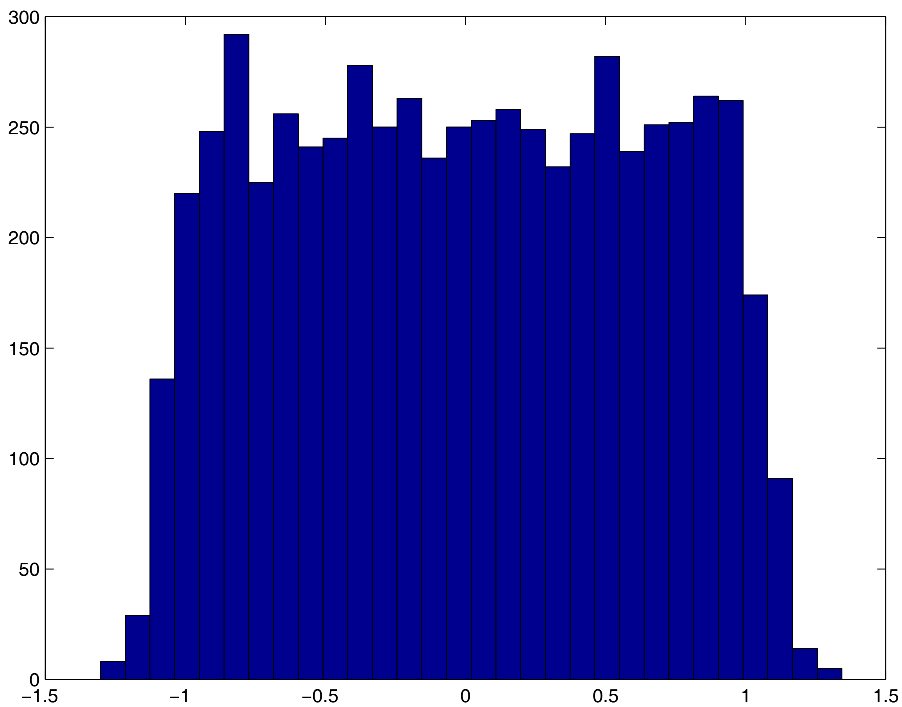
(real matrices)



Normalized eigenvalues of 20 real matrices of size $n = 100$ represented by dots. Matrix entries are "drawn" independently from $N(0, 1)$.

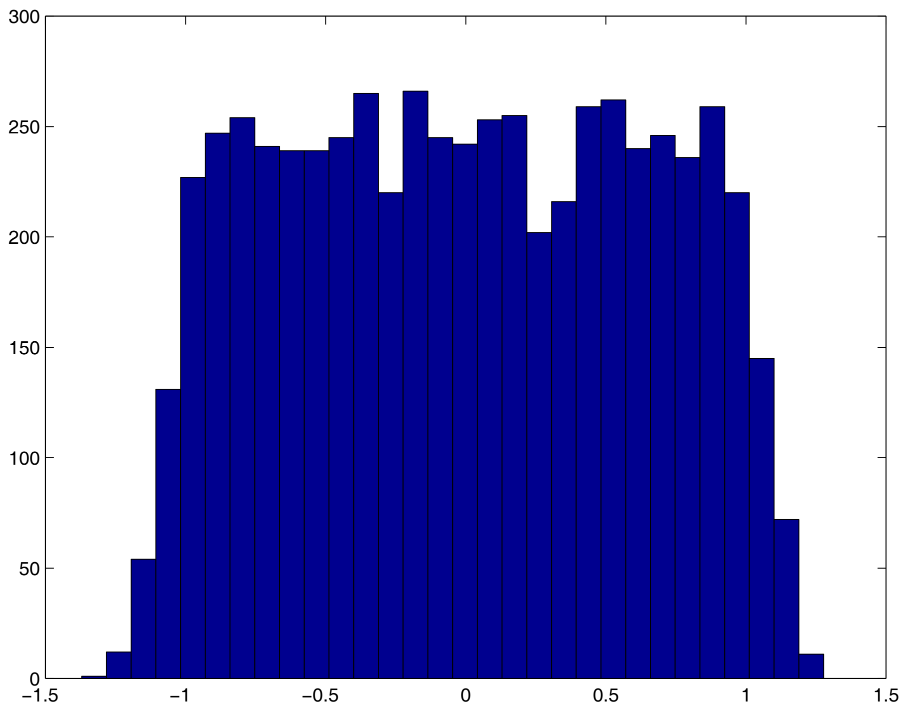
Uniform distribution of real eigenvalues

(real matrices with normally distributed entries)



Histogram of normalized real eigenvalues of 1000 real matrices of size $n = 50$. Matrix entries are "drawn" independently from $N(0,1)$. The total number of real eigenvalues is 6250.

Uniform distribution of real eigenvalues
(real matrices with uniformly distributed entries)



Histogram of normalized real eigenvalues of 1000 real matrices of size $n = 50$. Matrix entries are "drawn" independently from the uniform distribution with zero mean and unit variance. The total number of real eigenvalues is 6004.

Part II Weakly Non-Hermitian Random Matrices

Consider random $n \times n$ matrices $\tilde{J} = A + ivB$

(i) A and B are independent *Hermitian*, with i.i.d. entries
(ii) $E(A) = 0$, $E(B) = 0$
(iii) $E(\text{tr}\, A^2) = E(\text{tr}\, B^2) = \sigma^2 n^2$

Motivation: for any complex J
$J = X + iY$ where $X = \frac{J+J^*}{2}$ and $Y = \frac{J-J^*}{2i}$.

Since A and B are Hermitian, have $\tilde{J}_{kl}$ and $\tilde{J}_{lk}$ correlated for all $1 \leq k < l \leq n$:

$$E(\tilde{J}_{kl}\tilde{J}_{lk}) = E(|A_{kl}|^2) - v^2 E(|B_{kl}|^2) = \sigma^2(1 - v^2).$$

All other pairs are independent.

Have central matrix distribution with two parameters:

$$\sigma^2(1 + v^2) = E(|\tilde{J}_{kl}|^2)$$

and

$$\tau = \text{corr}(\tilde{J}_{kl}\tilde{J}_{lk}) = \frac{E(\tilde{J}_{kl}\tilde{J}_{lk})}{\sqrt{E(|\tilde{J}_{kl}|^2)E(|\tilde{J}_{lk}|^2)}} = \frac{1 - v^2}{1 + v^2}.$$

Without loss of generality, assume $\sigma^2 = 1/(1 + v^2)$, so that

$$E(|\tilde{J}_{kl}|^2) = 1 \text{ and } E(\tilde{J}_{kl}\tilde{J}_{lk}) = \tau$$

Typical eigenvalues of $\tilde{J}$ are of the order of $\sqrt{n}$, so introduce $J = \tilde{J}/\sqrt{n} = (A + ivB)/\sqrt{n}$.

Eigenvalue correlation functions $R_k^n(z_1, \ldots z_k)$:

$R_1^n(z)$ is the probability *density* of finding an eigenvalue of $J = \frac{\tilde{J}}{\sqrt{n}}$, *regardless of label*, at z.

E.g., if D_0 is an infinitesimal circle covering z_0, then the probability of finding an eigenvalue of J in D_0 is approximately $R_1^n(z_0) \times \text{area}(D_0)$.

Similarly, $R_k^n(z_1, \ldots z_k)$ is the *probability density* of finding an eigenvalue J, *regardless of labeling*, at each of the points $z_1, \ldots z_k$.

Have k slots $z_1, \ldots z_k$ and n eigenvalues of J to fill these slots, hence normalization:

$$\int \ldots \int R_k^n(z_1, \ldots z_k) d^2 z_1 \cdots d^2 z_k = n(n-1) \cdots (n-k+1).$$

$R_1^{(n)}(z)$ gives the mean density of eigenvalues at z, i.e.

$$R_1^{(n)}(z) = E\Big(\sum \delta^{(2)}(z - \lambda_j) \Big)$$

where the summation is over all eigenvalues λ_j of J and $\delta^{(2)}(x + iy) = \delta(x)\delta(y)$.

If N_D is the number of eigenvalues in D, then

$$E(N_D) = \int_D R_1^{(n)}(z)\, d^2 z = \int\!\!\int_D R_1^{(n)}(x, y)\, dx dy$$

Convention: $z = x + iy \equiv (x, y)$ and $d^2 z = dx dy$.

From now on, replace (i)-(iii) by

(iv) Hermitian A and B are drawn independently from the normal matrix distribution with density

$$\frac{1}{Q}\exp\left(-\frac{1}{2\sigma^2}\operatorname{tr}X^2\right)=\frac{1}{Q}\exp\left(-\frac{1}{2\sigma^2}\sum_{k,l=1}^{n}|X_{kl}|^2\right),$$

where $\sigma^2(1+v^2)=1$ (with no loss of generality).

Have

$$\begin{aligned} X_{kl} &\sim N\Big(0,\frac{1}{2}\sigma^2\Big)+i\times\text{indp.}\,N\Big(0,\frac{1}{2}\sigma^2\Big), \qquad k<l \\ X_{kk} &\sim N(0,\sigma^2) \end{aligned}$$

and the $\{X_{kl}\}$, $1\le k\le l\le n$ are independent.

The entries of $\tilde{J}=A+ivB$ have multivariate complex normal distribution with density

$$\exp\left[-\frac{1}{1-\tau^2}\Big(\operatorname{tr}\tilde{J}\tilde{J}^*-\frac{\tau}{2}\operatorname{Re}\operatorname{tr}\tilde{J}^2\Big)\right],\quad \tau=\frac{1-v^2}{1+v^2}.$$

Have $E(\tilde{J}_{kl})=0$ and $E(|\tilde{J}_{kl}|^2)=1$ for all (k,l) and

$$\begin{aligned} E(\tilde{J}_{kl}\tilde{J}_{mj}) &= \tau \quad \text{when}\, k=j \text{ and } l=m \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

- If $\tau=0$, then $\tilde{J}$ has independent entries (Ginibre's ensemble); have maximum asymmetry.

- If $\tau=1$ or $\tau=-1$, then $\tilde{J}=\tilde{J}^*$ (GUE) or $\tilde{J}=-\tilde{J}^*$, have no asymmetry at all.

Hermite polynomials:

$$H_n(z) = (-1)^n \exp\left(\frac{z^2}{2}\right) \frac{d^n}{dz^n} \exp\left(-\frac{z^2}{2}\right)$$

Generating function: $$\exp\left(zt - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}.$$

By making use of generating function,

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) \exp\Big(-\frac{x^2}{2}\Big)\, dx = \delta_{n,m} n! \sqrt{2\pi} \qquad (1)$$

and, for all $0 < \tau < 1$,

$$\frac{\tau^n}{\sqrt{1-\tau^2}} \int H_n\Big(\frac{z}{\sqrt{\tau}}\Big) H_n\Big(\frac{\bar{z}}{\sqrt{\tau}}\Big) w_\tau^2(z,\bar{z}) d^2 z = \delta_{n,m} \pi n! \qquad (2)$$

$$\begin{aligned} w_\tau^2(z,\bar{z}) &= \exp\Big\{-\frac{1}{1-\tau^2}\Big[|z|^2 - \frac{\tau}{2}(z^2+\bar{z}^2)\Big]\Big\} \\ &= \exp\Big(-\frac{x^2}{1+\tau} - \frac{y^2}{1-\tau}\Big) \end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) \to \delta(y), \quad \text{as } \sigma \to 0,$$

(1) can be obtained from (2) by letting $\tau \to 1$.

Useful integral representation:

$$H_n(z) = \frac{(\pm i)^n}{\sqrt{2\pi}} \exp\Big(\frac{z^2}{2}\Big) \int_{-\infty}^{+\infty} t^n \exp\Big(-\frac{t^2}{2} \mp izt\Big)\, dt.$$

Finite matrices

Theorem* *Under assumption (iv), for any finite* n *and any* $0 \le \tau \le 1$,

$$R_k^{(n)}(z_1, \dots z_k) = \det ||K_\tau^{(n)}(z_m, \bar{z}_l)||_{m,l=1}^k,$$

where

$$K_\tau^{(n)}(z_1, \bar{z}_2) = \frac{n}{\pi\sqrt{1-\tau^2}} \sum_{j=0}^{n-1} \frac{\tau^n}{j!} H_j\Big(\sqrt{\frac{n}{\tau}} z_1\Big) H_j\Big(\sqrt{\frac{n}{\tau}} \bar{z}_2\Big) \times \\ \exp\Big[-\frac{n}{2(1-\tau^2)} \sum_{j=1}^{2} (|z_j|^2 - \tau \operatorname{Re} z_j^2)\Big]$$

Special cases: $\tau = 0$ (Ginibre's ens.) and $\tau = 1$ (GUE).

When $\tau = 0$ (in the limit $\tau \to 0$, to be more precise):

$$K_0^{(n)}(z_1, \bar{z}_2) = \frac{n}{\pi} \sum_{j=0}^{n-1} \frac{n^j}{j!} z_1^j \bar{z}_2^j \exp\Big[-\frac{n}{2}(|z_1|^2 + |z_2|^2)\Big].$$

Can be seen from

$$\sqrt{\tau^j} H_j\Big(\frac{z}{\sqrt{\tau}}\Big) = z^n + \sqrt{\tau} \times (\dots)$$

Sketch of proof: obtain induced density of eigenvalues and use the orthogonal polynomial technique; the required orthogonal polynomials are Hermite polynomials $H_j\Big(\sqrt{\frac{1}{\tau}} z\Big)$, they are orthogonal in $\mathbf{C}$ with weight function $w^2(z, \bar{z})$

Mean eigenvalue density for finite matrices

By Theorem (*), $R^{(n)}(z) = K_\tau^{(n)}(z, \bar{z})$, and

(a) if $0 < \tau < 1$ then

$$R_1^{(n)}(z) = \frac{n}{\pi\sqrt{1-\tau^2}}\, e^{-n\frac{|z|^2 - \tau \operatorname{Re} z_j^2}{2(1-\tau^2)}} \sum_{j=0}^{n-1} \frac{\tau^n}{j!} \left| H_j\left(\sqrt{\frac{n}{\tau}} z\right) \right|^2 .$$

By letting $\tau \to 0$ in (a):

(b) If $\tau = 0$ (Ginibre's ensemble) then

$$R_1^{(n)}(z) = \frac{n}{\pi}\, e^{-n|z|^2} \sum_{j=0}^{n-1} \frac{n^j |z|^{2j}}{j!}.$$

By letting $\tau \to 1$ in (a):

(c) if $\tau = 1$ (GUE) then

$$R_1^{(n)}(z) \equiv R^{(n)}(x, y) = \delta(y) \sqrt{\frac{n}{2\pi}} e^{-\frac{n}{2}x^2} \sum_{j=0}^{n-1} \frac{1}{j!} \left| H_j(\sqrt{n}x) \right|^2 .$$

Limit of infinitely large matrices

Consider matrices $\tilde{J} = X + iY$.

Can have two regimes when $n \to \infty$:

- strong non-Hermiticity $E(\text{tr}\, Y^2) = O(E(\text{tr}\, X^2))$,
- weak non-Hermiticity $E(\text{tr}\, Y^2) = o(E(\text{tr}\, X^2))$.

If $v^2 > 0$ stays constant as $n \to \infty$, have strongly non-Hermitian $J = \frac{1}{\sqrt{n}}(A + ivB)$.

Recall $\tau = \frac{1-v^2}{1+v^2}$. The following result is a corollary of Theorem (*):

Theorem *(Girko's Elliptic Law) For any $\tau \in (-1, 1)$ and any bounded $D \subset \mathbf{C}$*

$$E(N_D) = n \iint_D \rho(x,y)\, dxdy + o(n)$$

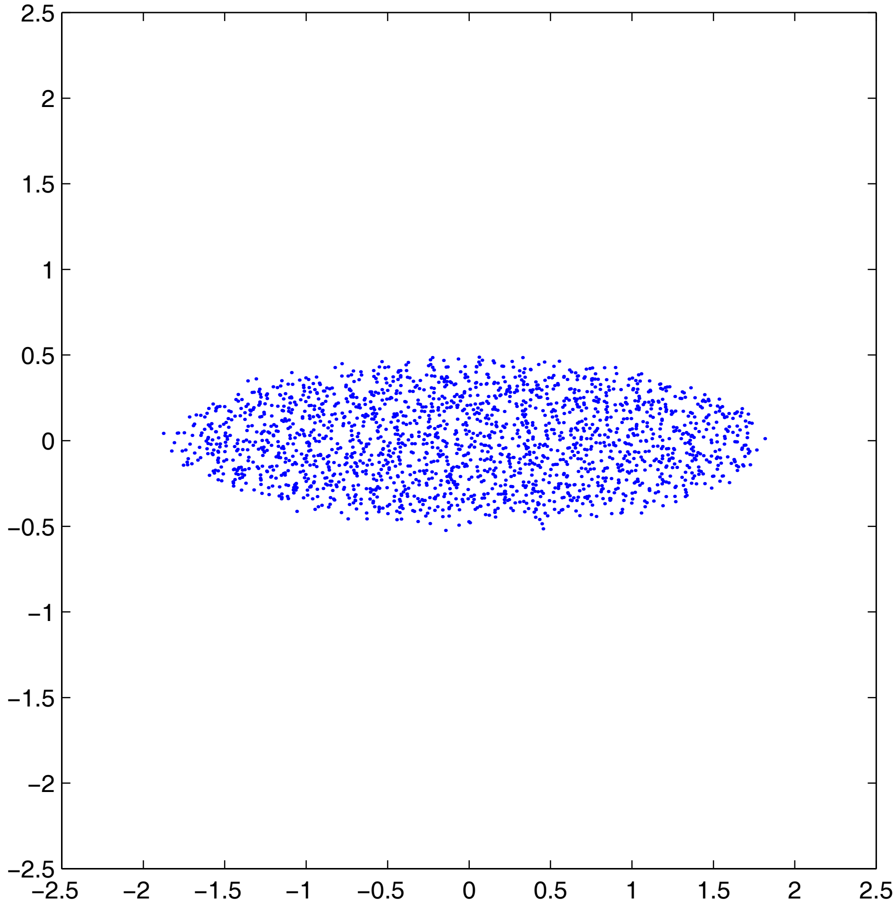
where N_D is the number of eigenvalues of J in D and

$$\rho(x,y) = \begin{cases} \frac{1}{\pi(1-\tau^2)}, & \text{when } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(Girko considered matrices J with symmetric pairs (J_{12}, J_{21}), (J_{13}, J_{31}), ... drawn independently from a bivariate distribution (not necessarily normal))

Note: $\lim_{\tau\to 1} \lim_{n\to\infty} \neq \lim_{n\to\infty} \lim_{\tau\to 1}$.

Elliptic distribution of eigenvalues



Eigenvalues of 20 **complex matrices** $J = \frac{A}{\sqrt{n}}A + iv\frac{B}{\sqrt{n}}$ of size $n = 100$ represented by dots. Matrices A and B are "drawn" independently from the GUE with normalization $E(\text{tr}\, A^2) = E(\text{tr}\, B^2) = n^2$ and $v = 0.5$.

Local scale: area is measured in units of mean density of eigenvalues, i.e. unit area contains, on average, 1 eigenvalue.

Unit area on the global scale is n times unit area on the local scale.

Limit distribution of eigvs of J: uniform in the ellipse

$$\mathcal{E} = \left\{ z : \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1 \right\}$$

of area $|\mathcal{E}| = \pi\sqrt{1-\tau^2}$. That is

$$E(N_D) \doteq \frac{|D \cap \mathcal{E}|}{|\mathcal{E}|}.$$

E.g. if $z_0 = x_0 + iy_0 \in \mathcal{E}$ and

$$D = \{z : |x - x_0| \leq \frac{\alpha}{2\sqrt{n|\mathcal{E}|}}, |y - y_0| \leq \frac{\beta}{2\sqrt{n|\mathcal{E}|}}\}$$

then $E(N_{D_0}) \doteq \alpha\beta$.

But also

$$E(N_D) = \int\int_D R_1^{(n)}(z) d^2 z = \int\int \frac{1}{n|\mathcal{E}|} R_1^{(n)}\Big(z_0 + \frac{w}{\sqrt{n|\mathcal{E}|}}\Big) d^2 w$$

Rescaled mean density of eigenvalues (around z_0):

$$\frac{1}{n|\mathcal{E}|} R_1^{(n)}\Big(z_0 + \frac{w}{\sqrt{n|\mathcal{E}|}}\Big)$$

Similarly, rescaled eigenvalue correlation functions:

$$\widehat{R}_k^{(n)}(w_1,\dots,w_k) := \frac{1}{(n|\mathcal{E}|)^k}\, R_k^{(n)}\Big(z_0+\frac{w_1}{\sqrt{n|\mathcal{E}|}},\ \dots,\ z_0+\frac{w_k}{\sqrt{n|\mathcal{E}|}}\Big)$$

The following result is a corollary of Theorem (*):

Theorem *For any $\tau \in (-1,1)$ and $z_0 \in$ int$\mathcal{E}$*

$$\lim_{n\to\infty} \widehat{R}_k^{(n)}(w_1,\dots,w_k) = \det ||K(w_m,\bar{w}_l)||_{m,l=1}^k,$$

where

$$K(w_1,\bar{w}_2) = \exp\Big(w_1\bar{w}_2 - \frac{1}{2}|w_1|^2 - \frac{1}{2}|w_2|^2\Big)$$

E.g., the first two correlation fncs:

$$\widehat{R}_1(w) = K(w,\bar{w}) = 1$$

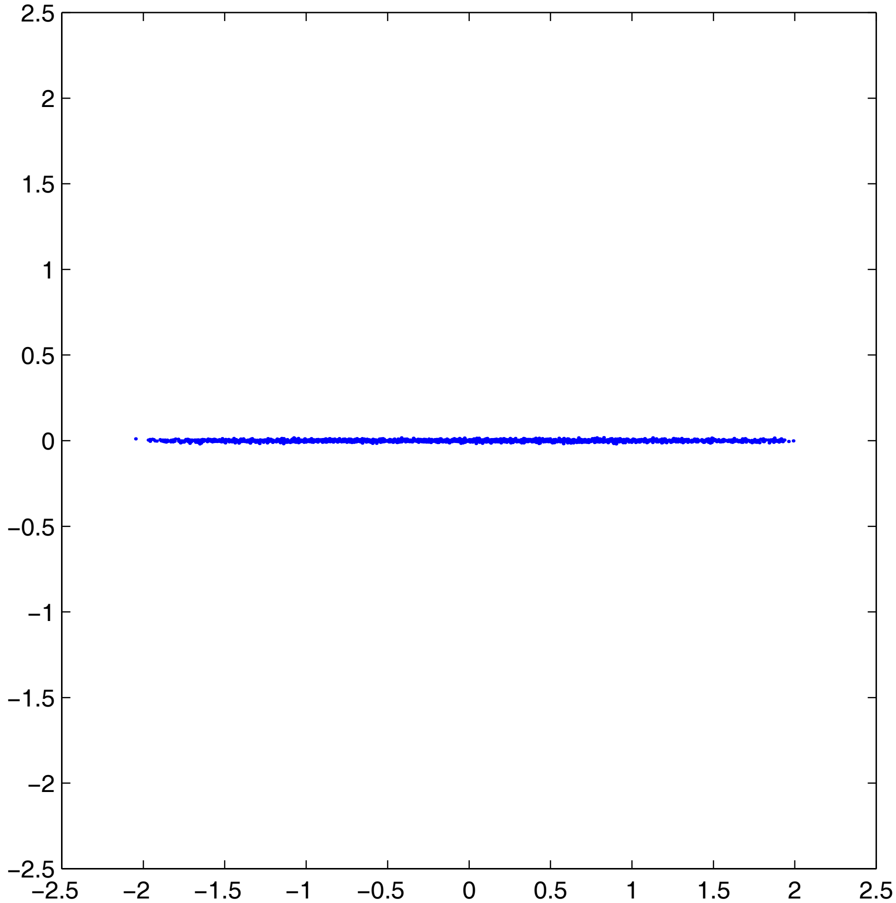
$$\begin{aligned}\widehat{R}_2(w,w_2) &= \widehat{R}_1(w_1)\widehat{R}_1(w_2) - |K(w_1,\bar{w}_2)|^2 \\ &= 1 - \exp\Big(-|w_1-w_2|^2\Big).\end{aligned}$$

No dependence on z_0, and, remarkably, no dependence on τ.

Again,

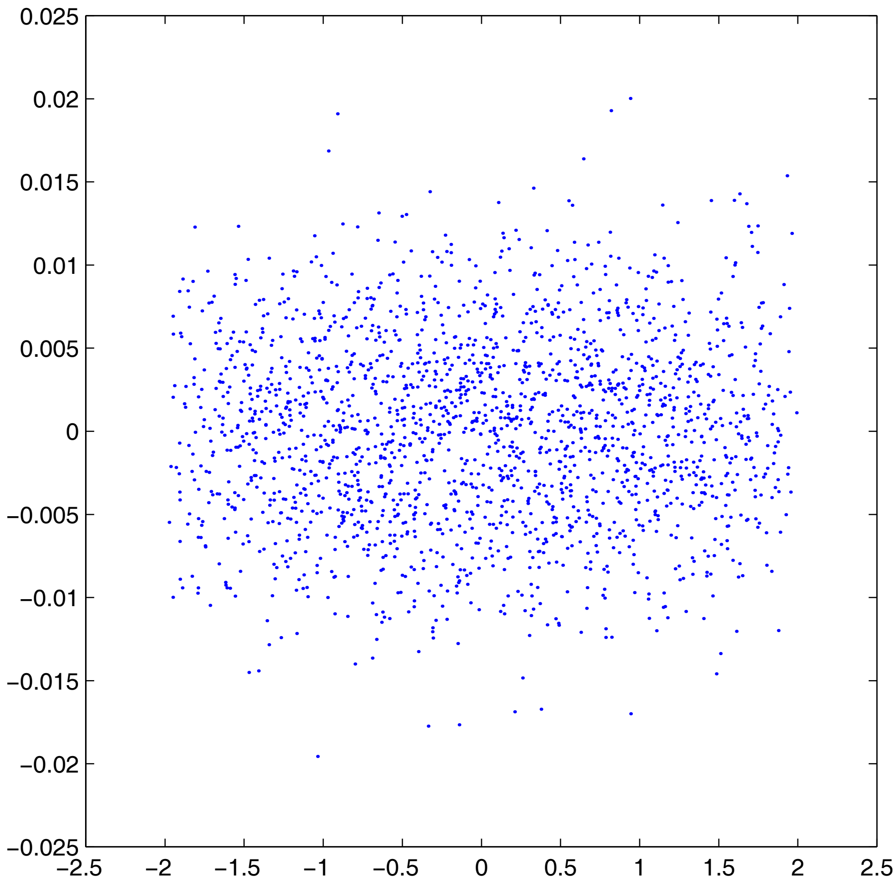
$$\lim_{\tau\to 1}\lim_{n\to\infty} \neq \lim_{n\to\infty}\lim_{\tau\to 1}.$$

Eigenvalues of weakly non-Hermitian matrices



Eigenvalues of 20 **complex matrices** $J = \frac{A}{\sqrt{n}}A + iv\frac{B}{\sqrt{n}}$ of size $n = 100$ represented by dots. Matrices A and B are "drawn" independently from the GUE with normalization $E(\text{tr}\, A^2) = E(\text{tr}\, B^2) = n^2$ and $v = 0.05$.

Eigenvalues of weakly non-Hermitian matrices



Eigenvalues of 20 **complex matrices** $J = \frac{A}{\sqrt{n}}A + iv\frac{B}{\sqrt{n}}$ of size $n = 100$ represented by dots. Matrices A and B are "drawn" independently from the GUE with normalization $E(\mathrm{tr}\, A^2) = E(\mathrm{tr}\, B^2) = n^2$ and $v = 0.05$.

Regime of weak non-Hermiticity

Now consider matrices $J = \frac{A}{\sqrt{n}} + iv\frac{B}{\sqrt{n}}$ in the limit when

$$n \to \infty \quad \text{and} \quad v^2 n \to \text{ const.} \tag{3}$$

May think of eigenvalues of J as of perturbed eigenvalues of $\frac{A}{\sqrt{n}}$. The eigenvalues of $\frac{A}{\sqrt{n}}$ are all real and are distributed in $[-2,2]$ with density

$$\nu_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2} \quad \text{(Wigner's semicircle law!)}$$

When perturbed they move off $[-2,2]$ into $\mathbf{C}$ on the distance of the order $\frac{1}{n}$ (first order perturbations). Correspondingly, consider

$$D = \{(x,y) : x \in I \subset [-2,2], \frac{s}{n} \le y \le \frac{t}{n}\}.$$

Then

$$E(N_D) = \int\int_D R_1^{(n)}(x,y)dxdy = \int_I dx \int_s^t d\hat{y}\, \frac{1}{n} R_1^{(n)}\left(x, \frac{\hat{y}}{n}\right),$$

where

$$\hat{y} = ny.$$

Hence

$$\hat{\rho}^{(n)}(x,\hat{y}) := \frac{1}{n^2} R_1^{(n)}\left(x, \frac{\hat{y}}{n}\right)$$

is the mean density of rescaled (distorted) eigenvalues $\hat{z} = x + i\hat{y} = x + iny$.

The following result is a corollary of Theorem (*).

Theorem *(Fyodorov, Khoruzhenko and Sommers)*

Let $\tau = 1 - \frac{\alpha^2}{2n}$. *Then, under assumption (iv),*

$$\lim_{n\to\infty} \widehat{\rho}^{(n)}(x,\widehat{y}) = \widehat{\rho}(x,\widehat{y}),$$

where

$$\widehat{\rho}(x,\widehat{y}) = \frac{1}{\pi\alpha}\,\mathrm{exp}\left(-\frac{2\widehat{y}^2}{\alpha^2}\right)\int_{-\pi\nu_{sc}(x)}^{\pi\nu_{sc}(x)} \mathrm{exp}\left(-\frac{\alpha^2u^2}{2} - 2u\widehat{y}\right)\frac{du}{\sqrt{2\pi}}.$$

In the limit when $\alpha \to 0$

$$\frac{1}{\sqrt{2\pi}\pi\alpha}\,\mathrm{exp}\left(-\frac{2\widehat{y}^2}{\alpha^2}\right) \to \frac{1}{2\pi}\delta(\widehat{y})$$

and

$$\widehat{\rho}(x,\widehat{y}) \to \delta(\widehat{y})\nu_{sc}(x) \qquad \text{Wigner's semicircle law}$$

Introduce curvilinear coordinates in the $(x, \widehat{y})$ plane:

$$(x, \tilde{y}) = \Big(x, \frac{\widehat{y}}{\pi\nu_{sc}(x)}\Big).$$

If

$$\tilde{\rho}(x, \tilde{y}) = \frac{1}{\pi\nu_{sc}(x)}\widehat{\rho}\Big(x, \frac{\widehat{y}}{\pi\nu_{sc}(x)}\Big)$$

then

$$\tilde{\rho}(x, \tilde{y}) = \nu_{sc}(x) p_x(\tilde{y}),$$

where

$$p_x(\tilde{y}) = \frac{1}{\sqrt{2\pi}a} \exp\Big(- \frac{a^2\tilde{y}^2}{2}\Big) \int_{-1}^{1} \exp\Big(- \frac{a^2\tilde{y}^2}{2} - 2t\widehat{y}\Big)\frac{dt}{\sqrt{2\pi}}$$

and $a = \pi\nu_{sc}(x)\alpha$.

- Interpretation of $p_x(\tilde{y})$.
- Universality of $p_x(\tilde{y})$.

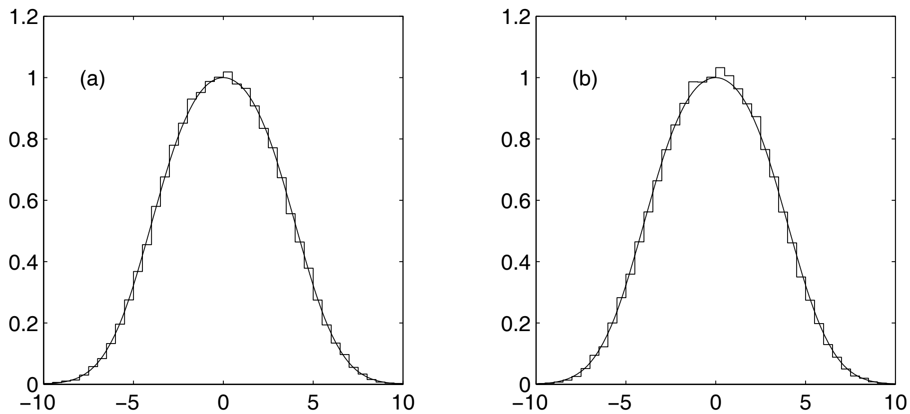
In the limit when $a \to \infty$ obtain uniform density

$$\tilde{\rho}(x, \tilde{y}) \doteq \begin{cases} \frac{1}{\pi a^2}, & \text{when } |\tilde{y}| \le \frac{a^2}{2} \\ 0, & \text{otherwise} \end{cases}$$

Eigenvalue correlation functions:

have a crossover from Wigner-Dyson to Ginibre

Weakly non-Hermitian matrices



Histogram of the scaled imaginary parts $\tilde{y}$ of complex eigenvalues of weakly non-Hermitian matrices
$J = \frac{A}{\sqrt{n}} A + iv\frac{B}{\sqrt{n}}$ of size $n = 30$. $v = \frac{1}{\sqrt{n}}$

The solid line is the graph of $p_x(\tilde{y})$ $(n = \infty)$.

For each plot 20000 matrices were generated and diagonalized. Eigenvalues $z_j = x_j + iy_j$ falling into the window, $|x_j| \leq 0.2$, were selected and their imaginary parts y_j were scaled, $\tilde{y}_j = 2\pi\nu_{sc}(0)ny_j$.

For plot (a), the matrices A and A were "drawn" independently from the GUE with normalization $E(\text{tr}\, A^2) = E(\text{tr}\, B^2) = n^2/2$.

For plot (b), the entries of A and B were "drawn" from Bernoulli($\frac{1}{2}$).

Another type of weakly non-Hermitian matrices:

- Dissipative matrices:
  $J = A + i\Gamma$, $\Gamma \geq 0$ and is of finite rank m

Weakly non-unitary matrices:

- Submatrices of size m of unitary matrices of size n, in the limit $n \to \infty$ and $m = n - a$, a is a constant.

- Contractions: random matrices $J = U\sqrt{I-T}$, where $U \in U(n)$ and $0 \leq T \leq I$ in the limit when $n \to \infty$ and the rank of T remains finite. (Note that $J^*J = I - T$)

Weakly asymmetric matrices

- $J = A + vB$, where A and B are real and $A^T = A$, $B^T = -B$.

Part III Asymmetric Tridiagonal Random Matrices

Imposing periodic boundary conditions:

$$\begin{pmatrix} & q_1 & b_1 & & & a_1 \\ & a_2 & q_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & b_n & & a_n & q_n & \end{pmatrix}$$

Problem: Fix a rectangle $K \subset \mathbf{C}$ and let $n \to \infty$. **What proportion of eigenvalues of J_n are in K?** [Eigenvalue distribution].

Example: $a_j = a$, $b_j = b$, $q_j = q$ for all k and $a, b, q \in \mathbf{R}$. The limit eigenvalue distribution is supported by the ellipse

$$\{(x, y) : x = q + (a + b)\cos p,\ y = (a - b)\sin p,\ \ p \in [0, 2\pi]\}$$

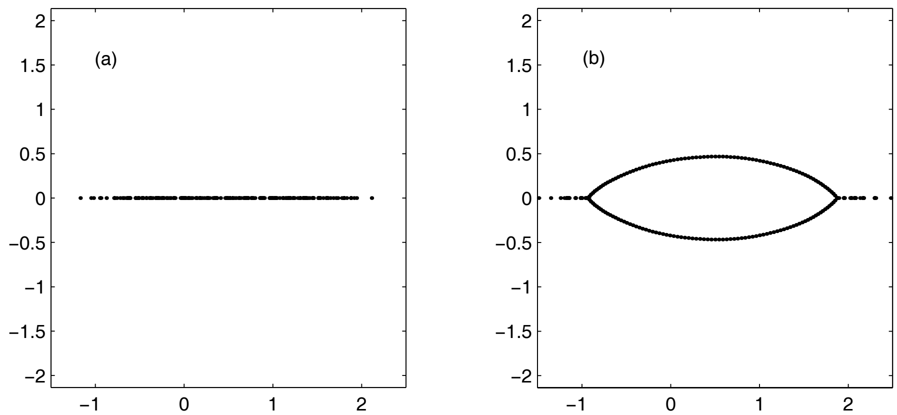
How will this picture change if allow random fluctuations of a_k, b_k and q_k? Answer depends on the sign of $a_k b_{k-1}$.

Consider

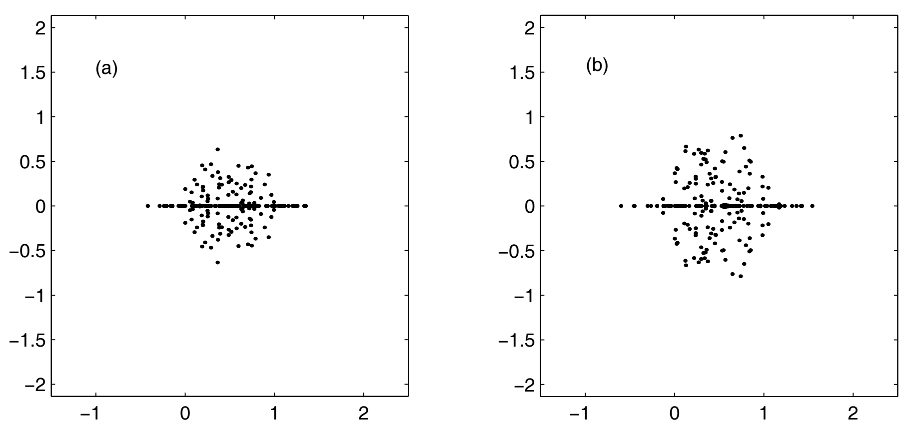
$$J_n = \text{tridiag}(a_k, q_k, b_k) \ \ + \ \ \text{p.b.c.}$$

with positive sub- and super-diagonals:

$$a_k = \exp(\xi_{k-1}),\ \ b_k = \exp(\eta_k)$$



Eigenvalues of J_n ($n = 201$) where (a) all non-zero entries are drawn from Uni$[0, 1]$; and (b) the sub-diagonal and diagonal entries are drawn from Uni$[0, 1]$ and super-diagonal entries are drawn from Uni$[\frac{1}{2}, 1\frac{1}{2}]$.



Eigenvalues of J_n ($n = 201$) where (a) the sub- and super-diagonal entries are drawn from Uni$[-\frac{1}{2}, \frac{1}{2}]$ and the diagonal entries are drawn from Uni$[0, 1]$; and (b) the sub-diagonal entries are drawn from Uni$[-\frac{1}{2}, \frac{1}{2}]$, and the diagonal and super-diagonal are drawn from Uni$[0, 1]$

Assumptions:

(I) (ξ_k, η_k, q_k), $k = 0, 1, 2, \ldots$, are independent samples from a probability distribution in $\mathbf{R}^3$.

(II) $E(\ln(1 + |q|))$, $E(\xi)$ and $E(\eta)$ are finite.

E.g. (ξ_k, η_k, q_k), $k = 0, 1, 2, \ldots$, are independent samples from a 3D prob. distr. with a compact supp. in $\mathbf{R}^3$.

By making use of the similarity transformation $W_n = \mathrm{diag}(w_1, \ldots w_n)$, $w_k = \exp\left[\frac{1}{2}\sum_{j=0}^{k-1}(\xi_j - \eta_j)\right]$,

$$W_n^{-1} J_n W_n = H_n + V_n,$$

where

$$H_n = \begin{pmatrix} q_1 & c_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & c_{n-1} & q_n \end{pmatrix} \quad V_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & u_n \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ v_n & 0 & \ldots & 0 & 0 \end{pmatrix}$$

$c_k = \sqrt{a_{k+1} b_k} = e^{\frac{1}{2}(\xi_k + \eta_k)}$ and

$$u_n / v_n = e^{n[\mathrm{E}(\xi_0 - \eta_0) + o(1)]} \qquad \text{as} \quad n \to \infty$$

rank 2 asymmetric perturb. of symmetric H_n!

"Rank 2" $\Rightarrow$ eignv. distbs. of H_n and H_n+V_n are related

"Strongly asymmetric" $\Rightarrow$ non-trivial relation.

Facts from theory of Hermitian random operators:

- Empirical distribution fnc. of eigvs. of H_n

$$N(I, H_n) = \frac{1}{n}\#\{\text{eigvs. of } H_n \text{ in } I \subset \mathbf{R}\}$$
$$= \int_I dN_n(\lambda), \quad N_n(\lambda) = N((-\infty, \lambda], H_n)$$

dN_n assigns mass $\frac{1}{n}$ to each of eigvs. of H_n.

Proposition $\exists$ *nonrandom* $N(\lambda)$ $\forall$ $I \subset \mathbf{R}$*:*

$$\lim_{n\to\infty} N(I, H_n) \overset{a.s.}{=} \int_I dN(\lambda)$$

- Potentials: $p(z; H_n) = \int \log|z - \lambda| dN_n(\lambda)$

$$\Phi(z) = \int \log|z - \lambda| dN(\lambda)$$

- Lyapunov exponent $\gamma(z) = \lim_{n\to\infty} \frac{1}{n} E(\ln ||S_n(z)||)$

Proposition *(Thouless formula)*

$$\lim_{n\to\infty} p(z; H_n) \overset{a.s.}{=} \Phi(z) \text{ unif. in } z \text{ on } K \subset \mathbf{C}\setminus\mathbf{R}$$
$$= \gamma(z) + \mathbf{E}\log c_0$$

Corollaries:

$\Phi(z)$ continuous in z;

$\Phi(x + iy) > \mathbf{E}\log c_0 \quad \forall y \neq 0;$ etc.

Consider

$$\mathcal{L} = \{z \in \mathbf{C} : \quad \Phi(z) = \max[E(\xi_0), E(\eta_0)]\}$$

This curve is an equipotential line of limiting eigenvalue distribution of H_n.

If the probability law of (ξ_k, η_k, q_k) has bounded support then $\mathcal{L}$ is confined to a bounded set in $\mathbf{C}$ and is a union of closed contours:

There are $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots$ such that

$$\mathcal{L} = \cup \mathcal{L}_j, \qquad \mathcal{L}_j = \{x \pm i y_j(x) : x \in [\alpha_j, \beta_j]\}$$

Notation:

$$N(K, J_n) = \frac{1}{n} \#\{\text{eigvs. of } J_n \text{ in } K\}, \qquad K \subset \mathbf{C}$$

(describes distribution of eigenvalues of J_n)

Theorem *(Goldsheid and Khoruzhenko) Assume (I-II). Then, with probability one,*

(a) $\forall K \subset \mathbf{C}\backslash\mathbf{R}$*:* $\quad N(K, J_n) \underset{n\to\infty}{\longrightarrow} \int\limits_{K\cap\mathcal{L}} \rho(z(s))ds$

where $\rho(z) = \frac{1}{2\pi}\left|\int \frac{dN(\lambda)}{z-\lambda}\right|$ *and* ds *is the arc-length measure on* $\mathcal{L}$*.*

(b) $\forall I \subset \mathbf{R}$*:* $\quad N(I, J_n) \underset{n\to\infty}{\longrightarrow} \int\limits_{I_W} dN(\lambda)$

where $I_W = I \cap \{\lambda : \ \Phi(\lambda + i0) > \max[E(\xi_0), E(\eta_0)]\}$

Sketch of proof: Let

$$p(z;J_n)=\frac{1}{n}\sum_{j=1}^{n}\log|z-z_j|=\frac{1}{n}\log|\det(J_n-z)|$$

where $z_1,\ldots,z_n$ are the eigenvalues of J_n.

Claim *(convergence of potentials)*

With probability one,

$$p(z;J_n)\underset{n\to\infty}{\longrightarrow}F(z)=\max[\Phi(z),E(\xi_0),E(\eta_0)]\qquad\forall z\notin\mathbf{R}\cup\mathcal{L}$$

The convergence is uniform in $z\in K\subset\mathbf{C}\setminus(\mathbf{R}\cup\mathcal{L})$.

Consider measures $d\nu_{J_n}$ assigning mass $\frac{1}{n}$ to each of the eigenvalues of J_n. Then

$$\frac{1}{2\pi}\Delta p(z;J_n)=d\nu_{J_n}$$

in the sense of distribution theory. By Claim, the potentials $p(z;J_n)$ converge for almost all $z\in\mathbf{C}$. This implies convergence in the sense of distribution theory. Since the Laplacian is continuous in $\mathcal{D}'$,

$$\frac{1}{2\pi}\Delta p(z;J_n)\to\frac{1}{2\pi}\Delta F(z)$$

in $\mathcal{D}'$. But then

$$d\nu_{J_n}\to d\nu\equiv\frac{1}{2\pi}\Delta F(z)$$

in the sense of of weak convergence of measures, hence Theorem.

Proof of Claim

$$\begin{aligned}\det(J_n - z) &= \det(H_n + V_n - z) \\ &= \det(H_n - z)\det(I_n + V_n(H_n - z)^{-1})\end{aligned}$$

Therefore

$$p(z; J_n) = p(z; J_n) + \frac{1}{n}\log|d_n(z)|.$$

V_n is rank 2. $V_n = A^T B$, where

$$A = \begin{pmatrix} u_n & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ v_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}\therefore\ d_n(z) &= \det(I_n + A^T B(H_n - z)^{-1}) \\ &= \det(I_2 + B(H_n - z)^{-1}A^T) \;\; 2 \times 2 \det \\ &= (1 + u_n G_{1n})(1 + v_n G_{n1}) - u_n v_n G_{11} G_{nn}\end{aligned}$$

where G_{lk} is the (k, l) entry of $(H_n - z)^{-1}$.

Now use

$$\begin{aligned}|u_n G_{1n}| &= e^{n[E(\xi_0) - \Phi(z) + o(1)]} \\ |v_n G_{n1}| &= e^{n[E(\eta_0) - \Phi(z) + o(1)]}\end{aligned}$$

and $|1 - u_n v_n G_{11} G_{nn}| \geq \alpha(z) > 0$, $z \notin \mathbf{R}$ to complete the proof.

Exactly solvable model

Consider $J_n = \text{tridiag}(e^g,\ \text{Cauchy}(0,b),\ e^{-g}) \ + \ \text{p.b.c.},$

$$\xi_k \equiv g, \quad \eta_k \equiv -g \quad P(q_k \in I) = \frac{1}{\pi}\int_I dq \frac{b}{q^2+b^2}$$

In this case $J_n = W_n^{-1}(H_n + V_n)W_n$, where

$$H_n = \text{tridiag}(1,\ \text{Cauchy}(0,b),\ 1) \qquad \text{Lloyd's model}$$

For Lloyd's model an explicit expression for $\Phi(z)$ is available:

$$4\cosh\Phi(z) = \sqrt{(x+2)^2 + (b+|y|)^2} + \sqrt{(x-2)^2 + (b+|y|)^2}$$

By making use of it,

- If $K = 2\cosh g \le K_{cr} = \sqrt{4+b^2}$ then $\mathcal{L}$ is empty.

- If $K > K_{cr}$ then $\mathcal{L}$ consists of two symmetric arcs

  $$y(x) = \pm\left[\sqrt{\frac{(K^2-4)(K^2-x^2)}{K^2}} - b\right] \quad -x_b \le x \le x_b$$

  x_b is determined by $y(x_b) = 0$.

Corollaries

$g = \frac{1}{2}E(\xi_0 - \eta_0)$ is a measure of asymmetry of J_n.

(1) Special case: Suppose that $q_k \equiv Const$ all k. Then $\gamma(0) = 0$ and $\gamma(z) > 0$ $\forall z \neq 0$. Since

$$\Phi(0) = \gamma(0) + \frac{1}{2}E(\xi_0 + \eta_0) < \max[E(\xi_0), E(\eta_0)]$$

the equation for $\mathcal{L}$, $\Phi(z) = \max[E(\xi_0), E(\eta_0)]$, has continuum of solutions for any $g \neq 0$.

For any $g \neq 0$ we have a bubble of complex eigv. around $z = 0$, i.e. no matter how small the perturb. V_n is, it moves a finite proportion of eigvs. of H_n off the real axis!

(2) Suppose now that the diagonal entries q_k are random. Then $\gamma(x) > 0$ $\forall x \in \mathbf{R}$ (Furstenberg) and

$$0 < \min_{x\in\Sigma} \gamma(x) = g_{\text{cr}}^{(1)} < g_{\text{cr}}^{(2)} = \max_{x\in\Sigma} \gamma(x) \leq +\infty$$

where Σ is the support of $dN(\lambda)$. Therefore

(a) If $|g| < g_{\text{cr}}^{(1)}$, J_n has zero proportion of non-real eigenvalues

(b) If $g_{\text{cr}}^{(1)} < |g| < g_{\text{cr}}^{(2)}$, J_n has finite proportions of real and non-real eigenvalues.

(c) $|g| > g_{\text{cr}}^{(2)}$, J_n has zero proportion of real eigenvalues.

A few elementary facts from potential theory.

Suppose that M_n is an $n \times n$ matrix. Denote by $d\nu_{M_n}$ the measure on $\mathbf{C}$ that assigns to each of the n eigenvalues of M_n the mass $\frac{1}{n}$. Its potential is given by

$$\begin{aligned} p(z; M_n) &= \frac{1}{n} \log |\det(M_n - zI_n)| \\ &= \int_{\mathbf{C}} \log |z - \zeta| d\nu_{M_n}(\zeta) \end{aligned}$$

$p(z; M_n)$ is locally integrable in z and for any sufficiently smooth function $f(z)$ with compact support

$$\begin{aligned} \int_{\mathbf{C}} \log |z - \zeta| \Delta f(z) d^2 z &= \lim_{\varepsilon \downarrow 0} \int_{|z-\zeta| \geq \varepsilon} \log |z - \zeta| \Delta f(z) d^2 z \\ &= 2\pi f(\zeta), \end{aligned}$$

by Green's formula. Hence

$$\frac{1}{2\pi} \int_{\mathbf{C}} p(z; M_n) \Delta f(z) d^2 z = \int_{\mathbf{C}} f(z) d\nu_{M_n}(z).$$

Both $p(z; M_n)$ and $d\nu_{M_n}$ define distributions in the sense of the theory of distributions and the equation above can be also read as the equality $d\nu_{M_n}(z) = \frac{1}{2\pi} \Delta p(z; M_n)$ where now Δ is the distributional Laplacian. More generally, it is proved in potential theory that, under appropriate conditions on $d\nu$, $d\nu(z) = \frac{1}{2\pi} \Delta p(z)$, where $p(z) = \int \log |z - \zeta| d\nu_{M_n}(\zeta)$ is the potential of $d\nu$. This Poisson's equation relates measures and their potentials.

Regularization of potentials

$$p_\varepsilon(z;J_n) = \frac{1}{2n}\log\det[(J_n - z)(J_n - z)^* + \varepsilon^2]$$

$$\begin{aligned}\frac{1}{2\pi}\Delta p_\varepsilon(z;J_n) &= \rho_\varepsilon(z;J_n)\\ &\underset{\varepsilon\to 0}{\rightarrow} \frac{1}{n}\sum \delta(z - z_j) \quad [n \text{ is finite}]\end{aligned}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} p_\varepsilon(z;J_n) = \lim_{n\to\infty}\lim_{\varepsilon\to 0} p_\varepsilon(z;J_n) \quad ??$$

Yes, for normal matrices. Counterexamples for non-normal matrices

In the vicinity of z_j:

$$\begin{aligned}\rho_\varepsilon(z;J_n) &\simeq \frac{(\kappa_j\varepsilon)^2}{\pi}\,\frac{1}{[(\kappa_j\varepsilon)^2 + |z - z_j|^2]^2}\\ &\underset{\varepsilon\to 0}{\rightarrow} \delta(z - z_j) \quad \text{if } \kappa_j \neq 0\end{aligned}$$

where $\kappa_j = |(\psi_j^L, \psi_j^R)^{-1}|$ and $\psi_j^{L(R)}$ are normalized left (right) eigevectors at z_j.

Spectral condition numbers, pseudospectra, etc.