# LECTURE 5

*Present Values of Annuities-Certain*

N.B. $\sum_{j=0}^{N-1} q^j = 1+q+q^2+\ldots+q^{N-1} = \dfrac{1-q^N}{1-q}$, for any $q \neq 1$. (1)

*Annuity* is a series of payments made at regular time intervals. Level annuity is a series of equal payments made at regular time intervals.

Two sorts of annuities: Annuity-Certain and Life-Annuity.

An annuity is certain if the *number* of payments is certain and specified in the contract. In contrast, the payments may depend on the survival of one or more human lives, then we say life-annuity. In this case the number of payments is uncertain (e.g. pensions are life annuities).

Consider $n$ payments of *one* unit of money to be made at intervals of one unit of time, the first is due one unit of time from now. This situation is known as an IMMEDIATE ANNUITY, the symbol for its present value is $a_{\overline{n}|}$ and

$$a_{\overline{n}|} = v+v^2+\ldots+v^n = v\frac{1-v^n}{1-v}.$$

These payments are made in arrear, i.e. at the end of each time period. If the same payment is made at the beginning of each time period, this is an ANNUITY-DUE, the symbol for its present value is $\ddot{a}_{\overline{n}|}$ and

$$\ddot{a}_{\overline{n}|} = 1+v+\ldots+v^{n-1} = \frac{1-v^n}{1-v}.$$

Notice that $a_{\overline{n}|} = v\,\ddot{a}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|} = 1+a_{\overline{n-1}|}$, and $a_{\overline{n}|} = \ddot{a}_{\overline{n+1}|} - 1$ (can you interpret these relations in words?).

$a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ are monotone increasing functions of $n$. Their limits, when $n \to \infty$,

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i} \qquad \text{and} \qquad \ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}.$$

give the present values of immediate annuity and annuity-due paid in *perpetuity*.

*Deferred Annuities*

Suppose a series of $n$ unit payments starts at time $m+1$, the last one due at time $m+n$. This may be considered as an *immediate annuity deferred m time periods*. The symbol for the present value is ${}_m|a_{\overline{n}|}$ and

$$\begin{aligned} {}_m|a_{\overline{n}|} = v^m+\ldots+v^{m+n} &= v^m a_{\overline{n}|} \\ &= a_{\overline{m+n}|} - a_{\overline{m}|} \end{aligned}$$

Deferred annuity-due: ${}_m|\ddot{a}_{\overline{n}|} = v^m\,\ddot{a}_{\overline{n}|}$

*Continuous Annuities*



This is a theoretical conception; in practice continuous annuities can be used to approximate frequent payments.

Suppose annuity payments are made continuously over $n$ units of time at rate 1 unit of money per unit time. The symbol for the present value is $\bar{a}_{\overline{n}|}$ and

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-t\delta}dt = \frac{1-v^n}{\delta}. \qquad (2)$$

We can also have deferred continuous annuity

$${}_m|\bar{a}_{\overline{n}|} = \int_m^{m+n} e^{-t\delta}dt = v^m\,\bar{a}_{\overline{n}|}.$$

*Annuities Payable p-thly*

Consider an annuity of *one unit of money per unit time* payable over $n$ units of time in installments of $\frac{1}{p}$ at $p$-thly intervals, the first payment due one $p$-thly interval from now (i.e. the payments are made in arrear). Its present value is denoted by the symbol $a^{(p)}_{\overline{n}|}$, so $a^{(12)}_{\overline{10}|}$ provides $\frac{1}{12}$ at the end of each month for 10 years.

Using (1),

$$a^{(p)}_{\overline{n}|} = \frac{1}{p}(v^{\frac{1}{p}}+v^{\frac{2}{p}}+\ldots+v^{\frac{np}{p}}) = \frac{v^{\frac{1}{p}}}{p}\left[\frac{1-v^n}{1-v^{\frac{1}{p}}}\right]$$

Likewise, $\ddot{a}^{(p)}_{\overline{n}|}$ is the symbol for annuity of one unit of money per unit time payable over $n$ units of time in installments of $\frac{1}{p}$ at $p$-thly intervals, the first payment due at time $t=0$ (i.e. the payments are made in advance).

Notice that the present value of annuity-due of 1 p.a. payable $p$-thly is the same the value at time $t=\frac{1}{p}$ of immediate annuity of 1 p.a. payable $p$-thly, hence

$$\ddot{a}^{(p)}_{\overline{n}|} = (1+i)^{\frac{1}{p}} a^{(p)}_{\overline{n}|} = \frac{1}{p}\left[\frac{1-v^n}{1-v^{\frac{1}{p}}}\right].$$

Also notice that $\bar{a}_{\overline{n}|}$ approximate both $\ddot{a}^{(p)}_{\overline{n}|}$ and $a^{(p)}_{\overline{n}|}$ in the limit when $p\to\infty$:

$$\lim_{p\to\infty} \ddot{a}^{(p)}_{\overline{n}|} = \bar{a}_{\overline{n}|} = \lim_{p\to\infty} a^{(p)}_{\overline{n}|}$$

The accumulated value of immediate annuity of 1 p.a. payable $p$-thly at the time the last installment is paid is denoted by the symbol $s^{(p)}_{\overline{n}|}$ ($\ddot{s}^{(p)}_{\overline{n}|}$ if annuity-due). Obviously, $a^{(p)}_{\overline{n}|} = v^n s^{(p)}_{\overline{n}|}$ and $\ddot{a}^{(p)}_{\overline{n}|} = v^n\,\ddot{s}^{(p)}_{\overline{n}|}$.