LECTURE 20

Whole-Life Annuities

A life annuity is a series of regular payments dependent on the survival of a life (x); no payments are to be made after the death of (x).

A whole-life annuity is payable until the death of (x).

n-year term life annuity is payable until the death of (x) for at most n years.

Life annuities can be classified according to the mode of a payment: (1) annually in advance, (2) $p$thly in advance, (3) $p$thly in arrears, (4) annually in arrears, and (5) continuously. (compare these modes with those in the table at the bottom of p. 11)

The expected present value (E.P.V.) of a whole-life annuity to be taken out by a life (x), payable at the rate of 1 unit of money per annum is denoted by

$\ddot{a}_x$	if the annuity is payable annually in advance,
$\ddot{a}_x^{(p)}$	if the annuity is payable $p$thly in advance,
$a_x^{(p)}$	if the annuity is payable $p$thly in arrears,
a_x	if the annuity is payable annually in arrears,
$\bar{a}_x$	if the annuity is payable continuously.

One can express E.P.V. of annuities in terms of life-table functions using the indicator of the survival of (x) to age $x+t$:

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } T(x) > t \\ 0 & \text{otherwise} \end{cases}$$

As the probability of the survival to age $x+t$ is ${}_tp_x$, $\mathbb{1}_t \sim$ Bernoulli(${}_tp_x$).

Annuities payable annually

The present value of a whole-life annuity payable annually at rate of 1 p.a. <u>in advance</u> can be written as

$$z = \sum_{k=0}^{\infty} v^k \mathbb{1}_k. \tag{1}$$

The first payment (of 1 unit of money) is due now, at the present time, and accordingly the first term in the sum above is 1. The kth term in the sum above is the present value of 1 payable on the survival of (x) to age $x+k$, hence the factor χ_k.

Taking the mathematical expectation

$$\ddot{a}_x = E(z) = E\left(\sum_{k=0}^{\infty} v^k \mathbb{1}_k\right) = \sum_{k=0}^{\infty} v^k E(\mathbb{1}_k) = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\infty} v^k \frac{l_{x+k}}{l_x} = \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x},$$

where we have used that $E(\mathbb{1}_k) = {}_kp_x$. Therefore,

$$\ddot{a}_x = \frac{N_x}{D_x}.$$

If we consider $z - 1 = \sum_{k=1}^{\infty} v^k \mathbb{1}_k$ this will give us the present value of a whole-life annuity payable annually at rate of 1 p.a. in arreas. For life annuities payable in arrears complemented by the corresponding payment at the present time become life annuities payable in advance.

$$a_x = E(z-1) = E(z) - 1 = \ddot{a}_x - 1 = \frac{N_x - D_x}{D_x} = \frac{N_{x+1}}{D_x}.$$

Annuities payable pthly

If an annuity is payable in regular installments p times per year at the rate of 1 p.a., then each installment is of $\frac{1}{p}$.

The present value of a whole life annuity payable $p$thly at rate 1 p.a. in advance is the sum of the present values of $\frac{1}{p}$ payable on the survival of x to ages $x + \frac{k}{p}$, $k = 0, 1, 2, \ldots$,

$$z = \sum_{k=0}^{\infty} \frac{1}{p} v^{\frac{k}{p}} \mathbb{1}_{\frac{k}{p}}$$

and

$$\ddot{a}_x^{(p)} = E(z) = \frac{1}{p} \sum_{k=0}^{\infty} v^{\frac{k}{p}} E(\mathbb{1}_{\frac{k}{p}}) = \frac{1}{p} \sum_{k=0}^{\infty} v^{\frac{k}{p}} {}_{\frac{k}{p}}p_x = \frac{1}{p} \sum_{k=0}^{\infty} v^{\frac{k}{p}} \frac{l_{x+\frac{k}{p}}}{l_x} = \frac{1}{p} \sum_{k=0}^{\infty} \frac{D_{x+\frac{k}{p}}}{D_x}$$

As the P.V. of a continuous flow of money at rate 1 p.a. equals the infinite-p limit of the present value of a series of $p$thly payments made at rate 1 p.a. over the same period of time, we obtain

$$\bar{a}_x = \lim_{p \to \infty} \ddot{a}_x^{(p)} = \lim_{p \to \infty} \frac{1}{p} \sum_{k=0}^{\infty} \frac{D_{x+\frac{k}{p}}}{D_x} = \int_0^{\infty} \frac{D_{x+t}}{D_x} dt = \int_0^{\infty} v^t \frac{l_{x+t}}{l_x} dt$$

Therefore, the expected present value of a whole-life annuity payable continuously at rate 1 p.a. is given by

$$\bar{a}_x = \int_0^{\infty} v^t \, {}_tp_x \, dt. \tag{2}$$