Queen Mary
University of London

**Examination by course units**
**22nd May 2006, 2.30-4.30**

**MAS224 Actuarial Mathematics** **Duration 2 hours**

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*Except for the award of a bare pass, only your best three questions will be counted. A sheet of formulae and certain life tables are provided for this examination.*

*Calculators ARE permitted in this examination, but no programming, graph plotting or algebraic facility may be used. The unauthorised use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the make and type of calculator used.*

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1. (a) State the relationship between the nominal rate of interest $i^{(p)}$ per annum when interest is compounded $p$-thly and the annual effective rate of interest $i$. Find the annual effective rate of interest if the nominal rate of interest is 8% per annum when interest is compounded monthly. How much interest should be paid in arrears for the use of £1000 over (i) a one month period? (ii) a one year period?

   (b) Matt Dillon invests £100 per month in arrears for ten years in a savings account. Find the accumulation at the end of the ten year period if the APR for the account is 10%.

   (c) State and interpret the relation between the annual rate of discount $d$ and the annual rate of interest $i$. Find the annual rate of discount if the APR is 12%. How much interest should be paid in advance for the use of £1000 over a period of (i) one year? (ii) three years?

   (d) John Brown is paying off a loan for £$10,000$ taken out exactly three years ago and being paid annually in arrears for 10 years. The third payment has just been made. The annual interest rate charged on the loan is 10%. Find the annual payment. Give the schedule of payments and obtain the amount of the loan which is still outstanding.

   If the interest changes to 8% per annum, but he continues with his original payment, how long will he take to pay off the loan. What will his final payment be?

2. Assume the mortality given in table ELT12.

   (a) Express the life table functions ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, $l_x$ and ${}_nd_x$ in terms of the survival function $S(t)$ and the radix $l_0$.

   (b) Calculate the probability that a life aged 21 dies before retirement at age 65.

   (c) Calculate the probability that a life aged 21 survives to age 65 but dies before age 66.

   (d) Calculate the expected number who survive to age 65 out of 1000 individuals aged 20.

   (e) When $x$ is an integer, use linear interpolation on $S(x+t)$ for $0 < t < 1$ to show that ${}_td_x \approxeq t \times {}_1d_x$. Calculate the number expected to die within six months after retirement at age 65 out of $100,000$ newborns.

   (f) Define the curtate further lifetime, $K(x)$, for a life aged $x$. For any integer $k \geq 0$, show that $P(K(x) = k) = \frac{S(x+k)-S(x+k+1)}{S(x)}$. Hence prove that $e_x \equiv E[K(x)] = \frac{1}{l_x}\sum_{r=1}^{\infty} l_{x+r}$.

   (g) Jack Russell retires through ill health at age 50. With the exception of the first two years after retirement, his mortality is the mortality given in table ELT12. The chance he survives the first year after retirement is $\frac{1}{2}p_{50}$ and the chance that he will then survive the following year is $\frac{3}{4}p_{51}$.

   Find $l_{[50]}$ and $l_{[50]+1}$. Derive Jack's complete non-curtate further expectation of life (conditional on survival): (i) at the time of his retirement; (ii) one year after retirement.

   You may assume the approximate relation that $\mathring{e}_x \approxeq e_x + \frac{1}{2}$.

3. Assume a 4% annual interest rate and the mortality given by table A1967-70 select values.

   (a) Find the cost of a 20-year endowment assurance policy, with equal death and endowment benefits of £$100,000$, taken out by a life aged 40. The death benefit is payable at the end of the year of death.

   (b) Calculate the annual premium required to purchase the policy in part (a) if the premium is to be paid annually in advance for 20 years contingent on survival.

   (c) Find the surrender value of the policy if the life assured surrenders the policy at age 50, just before the eleventh payment is due. How much fully paid-up whole life assurance would this provide?

   (d) Let $Y$ be the present value of an $n$-year life annuity due purchased by a life aged $x$ with payment of one unit made annually. Write $Y$ in terms of $K(x)$, the number of complete further years lived by a life aged $x$.

   Hence or otherwise show that $E[Y] = \frac{N_x - N_{x+n}}{D_x}$.

4. Let $X$ be the age-at-death of an individual, $S(x) = P(X > x)$ be the survival function and $\mu(x)$ be the instantaneous death rate. Also let $T(x)$ be the complete non-curtate further lifetime of a life aged $x$.

   (a) Express $P(T(x) > t)$ and $\mu(x)$ in terms of the survival function. Show that the density function of $T(x)$ is given by $f_{T(x)}(t) = \frac{\mu(x+t)S(x+t)}{S(x)}$ for $t > 0$.

   If the lifetime $X$ has survival function $S(x) = e^{-x/50}$ for $x \geq 0$, find the instantaneous death rate and the density function of $T(x)$. Find $\mathring{e}_x = E[T(x)]$.

   (b) Consider an $n$-year pure endowment policy with C units of endowment benefit taken out by a life aged $x$. Express the present value of the policy, $Y$, in terms of $T(x)$.

   Show that $E[Y] = \frac{CV^n S(x+n)}{S(x)}$ and $Var(Y) = \frac{C^2 V^{2n} S(x+n)}{S(x)} \left(1 - \frac{S(x+n)}{S(x)}\right)$, where $V$ is the discounting factor.

   A life assurance company withdraws benefit payments from a fund earning interest at rate 8% per annum. They have just sold a block of 20-year endowment policies each with endowment benefit of £$10,000$ for 100 newborns. Assume the mortality of the newborns is that of table ELT12.

   If $Y$ is the present value of one of these policies, calculate $E[Y]$ and $\sqrt{(}Var(Y))$.

   Calculate the minimum amount *£B per policy* that the company needs to invest in the fund so that there is a probability of approximately 0.95 that this will be sufficient to pay the benefits incurred by these 100 endowment policies. You may assume that the upper 5% point of the standard normal distribution is 1.6449.

5. Consider females in a population in which there is no immigration or emigration. Let $n_x(t)$ be the expected number of females aged $x$ in the population at time $t$. Females are assumed to have independent lifetimes and to reproduce independently. The probability that a female aged $x$ at time $t$ will survive to time $t+1$ is $p_x$ and the expected number of female offspring she will produce in time $t$ to $t+1$ is $b_x$.

Show that

$$\begin{aligned} n_0(t+1) &= \sum_{x=0}^{\infty} b_x n_x(t) \\ n_{x+1}(t+1) &= p_x n_x(t) \text{ for } x = 0, 1, 2, \ldots \end{aligned}$$

(a) Suppose that this system of equations has a solution $n_x(t) \equiv n_x$. Show that $n_x = n_0 S(x)$ for $x = 0, 1, \ldots$ and $\sum_{x=0}^{\infty} b_x S(x) = 1$, where $S(x)$ is the survival function.

An isolated population has equal birth and death rates and stable population size of $5,000$. The life experience is that of table ELT12, so that $n(x) = Al_x$ for a suitable scaling factor $A$. If $N(x)$ is the number aged $x$ or more in the population, then you may assume that $N(x) \approxeq Al_x \left(\frac{1}{2} + \mathring{e}_x\right)$. Find the expected number of births each year and the expected number aged under 21.

(b) Animals in a population die before they reach age 3. Females only produce offspring at age 2; the expected number of offspring is 2. The probability that a female survives her first year of life is $\frac{3}{4}$ and the probability that a one-year old female survives to age 2 is $\frac{2}{3}$.

Write down the Leslie matrix. Show that $\sum_{x=0}^{2} b_x S(x) = 1$.

If the population size is $N$, find the stable solution $n_x$ for $x = 0, 1, 2$ so that $n_x(t) \equiv n_x$ for all $t = 0, 1, 2, \ldots$

At time $t = 0$ the population contains only one-year old animals; there being 100 one-year-old females. Calculate the expected number of female animals in each of the age groups at times $t = 1$, $t = 2$ and $t = 3$. Will the population sizes for each age become stable over time? Justify your answer.

**A list of selected formulae are given below.**

Present values of annuities certain:

$$\ddot{a}_{\overline{n|}} = \frac{1-V^n}{1-V} \qquad a_{\overline{n|}} = V\,\ddot{a}_{\overline{n|}}$$

$$\ddot{a}^{(p)}_{\overline{n|}} = \frac{1}{p}\left[\frac{1-V^n}{1-V^{\frac{1}{p}}}\right] \qquad a^{(p)}_{\overline{n|}} = V^{\frac{1}{p}}\,\ddot{a}^{(p)}_{\overline{n|}}.$$

Expected present values of life annuities:

$$\ddot{a}_x = \frac{N_x}{D_x} \qquad \ddot{a}_{x:\overline{n|}} = \frac{N_x - N_{x+n}}{D_x} \qquad a_x = \ddot{a}_x - 1$$

$$\ddot{a}^{(p)}_x \simeq \ddot{a}_x - \frac{p-1}{2p} \qquad a^{(p)}_x \simeq a_x + \frac{p-1}{2p} \qquad a^{(p)}_x = \ddot{a}^{(p)}_x - \frac{1}{p}$$

Conversion relationships:

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

$$A_x = 1 - d\,\ddot{a}_x$$

$$A^{(p)}_x = 1 - d^{(p)}\,\ddot{a}^{(p)}_x$$

$$A^{(p)}_{x:\overline{n|}} = 1 - d^{(p)}\,\ddot{a}^{(p)}_{x:\overline{n|}}$$

**Turn over . . . Tables ELT12 and A1967-70 follow**