Spectral determinants of complex random matrices

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Spectral Determinants

Motivation: - complex eigenvalues

\[ \langle | \text{det}(zI - W)|^2 \rangle_W \] related to eigv. distr. of \( W \) (sometimes explicitly)

Example 1: Gaussian ensembles

\[ P(W_N, W_N^\dagger) \propto e^{-\text{tr}W_NW_N^\dagger} \]

Mean density \( \rho_N(x, y) \) of complex eigenvalues \( z = x + iy \):

\[ \rho_N(x, y) \propto e^{-|z|^2} \langle |\text{det}(zI_{N-1} - W_{N-1})|^2 \rangle_{W_{N-1}} \] (complex \( W \))

\[ \rho_N(x, y) \propto ye^{-(x^2-y^2)} \text{erfc}(y) \langle |\text{det}(zI_{N-2} - W_{N-2})|^2 \rangle_{W_{N-2}} \] (real \( W \))

but mean density of real eigvs of real matrices is prop. to mean absolute value of spectral determinant,

\[ \rho_N(x) \propto e^{-x^2} \langle |\text{det}(xI_{N-1} - W_{N-1})| \rangle_{W_{N-1}} \]

Eigenvalue corr. fnCs are expressed in terms of higher moments of spectral dets.

[Ginibre '64, Lehmann & Sommers '91, Edelman '93, Edelman, Kostlan & Shub '94]
Spectral Determinants

Example 2: Finite rank deviations from Hermiticity or unitarity. E.g.,

If \( W_N(\gamma) = R_N U_N \) where \( U_N \) is CUE and \( R_N = \text{diag}(\sqrt{1 - \tilde{\gamma}}, 1, \ldots, 1) \) (rank-one deviations from CUE) then

\[
\rho_N(x, y) = \frac{N-1}{\pi\gamma|z|^2} \left( \frac{\tilde{\gamma}}{\gamma} \right)^{N-2} \langle |\det(z I_{N-1} - W_{N-1}(\tilde{\gamma}))|^2 \rangle_{U_{N-1}},
\]

where \( \tilde{\gamma} = \frac{|z|^2 + \gamma - 1}{|z|^2} \). Note that \( \gamma = 1 \) corresponds to subunitary matrices (delete 1st row & column). In this case \( \tilde{\gamma} = \gamma = 1 \).

If \( W_N(\gamma) = H_N + i\Gamma_N \), where \( H_N \) is GUE and \( \Gamma_N = \text{diag}(\gamma, 0, \ldots, 0) \) (rank-one deviations from GUE), then

\[
\rho_N(x, y) = r_{N,\gamma}(x, y) \langle |\det(z I_{N-1} - W_{N-1}(\tilde{\gamma}))|^2 \rangle_{H_{N-1}}
\]

where \( \tilde{\gamma} = \gamma - y \).

[Fyodorov & K '99, Życzkowski K & Sommers '02, Fyodorov & Sommers '03]
Spectral Determinants

Previous examples are special. In general, one would recover the mean density of eigenvalues from the mean fractional (absolute) moments of the spectral dets

\[ G(x, y) = \langle |\text{det}(zI - W)|^{2s} \rangle_W \]

Because of the singularities, some sort of regularization might be desirable, e.g.,

\[ \langle |\text{det}(zI - W)(zI - W)^\dagger + \varepsilon I|^{2s} \rangle_W \]

Unfortunately, we can handle integer moments only, \( G(x, y) \) for integer \( s \).

Note that if the distribution of \( W \) is invariant then

\[ \langle |\text{det}(zI - W)|^{2s} \rangle_W = \left\langle \int_{U(N)} |\text{det}(zI - WU)|^{2s} dU \right\rangle_W \]

In view of this, we consider the class of matrices \( W = AU \) where \( A \) is fixed and \( U \) is chosen at random from the unitary group \( U(N) \). We shall see that the integration over \( U \) (the ‘angular’ part of \( W \)) reduces non-Hermitian problem (moments of the spectral determinants) to a Hermitian one.
Angular integrals

For any two $N \times N$ matrices $A$ and $B$

$$
\int_{U(N)} \det(I - AU)^m \det(I - U^\dagger B^\dagger)^n dU \propto 
\int_{\mathbb{C}^{n \times m}} \frac{\det(I + Q^\dagger Q \otimes B^\dagger A)}{\det(I + Q^\dagger Q)^{N+n+m}} dQ, \quad m, n \geq 1.
$$

The integration on the RHS is over rectangular matrices $Q$ of size $n \times m$.

If $A^\dagger A < I_N$ and $B^\dagger B < I_N$ then

$$
\int_{U(N)} \frac{dU}{\det(I - AU)^m \det(I - U^\dagger B^\dagger)^n} = \int_{Q^\dagger Q \leq I} \frac{d\rho_{N,n \times m}(Q)}{\det(I - Q^\dagger Q \otimes B^\dagger A)},
$$

$$
1 \leq m, n \leq N,
$$

where $d\rho_{N,n \times m}$ is the push-forward of the Haar measure under the truncation $U \mapsto Q$.

Note that if $N \geq n + m$ then $d\rho_{N,n \times m}(Q) \propto \det(I - Q^\dagger Q)^{N-m-n} dQ$ [Friedman & Mello ’85; also Neretin ’02, Fyodorov & Sommers ’03, Forrester ’06]
Schur function expansions and CFT

The above integration formulas can be proved by making use of either the Schur function expansions or Zirnbauer’s Colour-Flavour Transformation (Zirnbauer '96), and in fact these two approaches are equivalent. The equivalence comes in the form of another pair of integral identities where \( s_\lambda \) are Schur functions:

- **Fermionic case.** For integer \( N \geq 0 \)
  \[
  \int_{C^{n \times m}} \frac{s_\lambda(Q^\dagger Q)}{\det(I + Q^\dagger Q)^{N + m + n}} \, dQ = \text{const.} \frac{s_\lambda(I_m)s_\lambda(I_n)}{s_\lambda'(I_N)} \quad \text{(UIF)}
  \]

- **Bosonic case.** For integer \( N \geq n, m \)
  \[
  \int s_\lambda(Q^\dagger Q) d\rho_{N,n \times m}(Q) = \frac{s_\lambda(I_m)s_\lambda(I_n)}{s_\lambda(I_N)} \quad \text{(UIB)}
  \]

(UIB) is a corollary of the invariance of \( d\rho_{N,n \times m} \). (UIF) seems to be new.

(UIB) implies the bCFT via Schur functions expansions, and vice versa. We can only show that (UIF) is equiv. to the fCFT in a particular case (corresponding to \( \langle |\det(I + AU)|^2 \rangle_U \)).
Selberg-type integrals

(UIB) can also be obtained from the Selberg type integral \( (J^1_{1/\gamma}) \) are Jack polynomials

\[
\int_0^1 \cdots \int_0^1 J_{1/\gamma}^{1/(\lambda)}(x_1, \ldots, x_m) \prod_{j=1}^m x_j^{p-1}(1-x_j)^{q-1} \prod_{1 \leq i < j \leq m} |x_i - x_j|^{2\gamma} \prod_{j=1}^m dx_j 
\]

\[
= J_{1/\gamma}^{1/(\lambda)}(1_m) \prod_{i=1}^m \frac{\Gamma(i\gamma + 1)\Gamma(\lambda_i + p + \gamma(m-i))\Gamma(q + \gamma(m-i))}{\Gamma(1+\gamma)\Gamma(\lambda_i + p + q + \gamma(2m-i-1))}.
\]

evaluated by Kadell '88 (for \( J_{1/\gamma}^{1/(\lambda)} = s_{1/\lambda} \)), Kadell '97 (general), Yan '92, Kaneko '93.

\(< J_{1/\gamma}^{1/(\lambda)} > \) in the fermionic case yet to be evaluated for arbitrary \( \gamma \) which is an interesting open problem. Known cases \( \gamma = 1, 2 \).

\( \gamma = 1 \): (Schur functions) the integral in both cases, can be evaluated by reducing it to binomial determinants (Fyodorov & K '06).

\( \gamma = 1/2 \) (zonal polynomials): the integral was evaluated by Constantine '63 in the bosonic case and his calculation can be extended to the fermionic case.
Applications: Feinberg-Zee single ring theorem

Consider random matrices $W \in \mathbb{C}^{N \times N}$ with inv. matrix distr. $e^{-N \text{Tr} V(W^* W)} dW$. Note that joint pdf of eigenvalues is only known for the Ginibre ensemble ($V(t) = t$).

In view of unitary invariance,

$$
\langle |\det(zI - W)|^{2m} \rangle_W \propto \int_{\mathbb{C}^{m \times m}} \frac{\langle \det(|z|^2 I + Q^\dagger Q \otimes W^\dagger W) \rangle_W}{\det(I + Q^\dagger Q)^{N+2m}} (dQ)
$$

Thus, integration over the angular part of $W$ can be traded for an average over $m \times m$ matrices $Q$ - Jacobi ensemble. Advantage - now have Hermitian matrices $W^\dagger W$, can apply orthogonal polynomial technique, etc. Structure - Hankel determinants. Matrix elements are integrals involving orthogonal polynomials.

Also advantageous for small values of $m$. E.g., $m = 1$

$$
\langle |\det(zI - W)|^2 \rangle_W = (N + 1) \int_0^{+\infty} \frac{\langle \det(|z|^2 + tW^\dagger W) \rangle_W (1 + t)^{n+2}}{(1 + t)^{n+2}} dt
$$
Applications: Feinberg-Zee single ring theorem

\[ \langle \det(I|z|^2 + tW^\dagger W) \rangle_W \]
can be evaluated asymptotically for large \( N \) in terms of the eigv. distribution \( d\sigma(\lambda) \) of the Hermitian matrices \( W^\dagger W \), yielding

\[ \langle |\det(zI - W)|^2 \rangle_W = \exp[N\Phi(x, y) + o(N)] \]

\[ \Phi(x, y) = \begin{cases} 
\log |z|^2 & \text{if } |z| > m_1 = \int \lambda d\sigma(\lambda), \\
\int_0^\infty \log \lambda d\sigma(\lambda) & \text{if } 1/|z| > m_{-1} = \int \frac{d\sigma(\lambda)}{\lambda}, \\
|z|^2 + \int_0^\infty \log \frac{\lambda + t_0}{|z|^2 + t_0} d\sigma(\lambda) & \text{if } 1/m_{-1} < |z| < m_1
\end{cases} \]

where \( t_0 \) is the (unique) solution of \( \int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = \frac{1}{|z|^2 + t} \).

Strong self-averaging (Berezin '73)

\[ \lim_{N \to \infty} \left\langle \frac{1}{N} \log |\det(zI - W)|^2 \right\rangle_W = (\?) \lim_{N \to \infty} \frac{1}{N} \log \langle |\det(zI - W)|^2 \rangle_W \]

Yes for GUE (Berezin '73). Yes for Ginibre (by direct computation \( \Phi(x, y) = |z|^2 - 1 \)). Yes beyond Ginibre as \( \Delta \Phi \) agrees with mean eigv density of \( W \) found by Feinberg & Zee '97.
Regularised inverse determinants

\[ R_\varepsilon(A^* A) = \int_{U(N)} \frac{dU}{\det[(I + AU)(I + AU)^* + \varepsilon^2 I]^m} \]

Non-trivial even for \( m = 1 \). Direct application of bCFT runs into a problem (diverging integrals). Schur functions do not help. A deformed version of CFT helps. Expression for \( R_\varepsilon(A^* A) \) in the simplest case \( m = 1 \):

\[
\frac{N - 1}{2\pi i} \int_0^1 (1 - t)^{N-2} dt \int_{-\infty}^{+\infty} \frac{ds}{s} \det \left[ A^\dagger A + (\varepsilon^2 - t) I - i\varepsilon \sqrt{t} \left( s + \frac{1}{s} \right) I \right].
\]

If the eigenvalues \( a_j^2 \) of \( A^\dagger A \) are distinct then, in the limit \( \varepsilon \to 0 \), this integral can be evaluated:

\[-c_N(z) \log \varepsilon^2 + d_N(z) + O(\varepsilon),\]

\[
c_N(z) = (N - 1) \sum_{j=1}^{N} (1 - |z|^2 a_j^2)^{N-2} \theta(1 - |z|^2 a_j^2) \prod_{k \neq j} \frac{1}{|z|^2(a_k^2 - a_j^2)}
\]

where \( \theta \) is Heaviside’s step fnc. For \( \lambda_{min}(A^\dagger A) \leq \frac{1}{|z|^2} \leq \lambda_{max}(A^\dagger A) \) have log-singularity \( (c_N(z) \neq 0) \).
Conclusions

- moments of spectral determinants is an interesting object, various links to truncations of random unitary matrices, CFT, Selber-type integrals, Berezin reproducing kernels (Berezin '75)

- stochastic Horn problem (singular values \(\leadsto\) eigenvalues) for spectral determinants can be solved by two equivalent methods: Schur function expansions or CFT.

- Feinberg-Zee’s ring density reproduced (but not proved); have conjecture:

\[
\frac{1}{N} \langle \log \det \rangle = \frac{1}{N} \log \langle \det \rangle \quad \text{(strong non-Hermiticity)}
\]

- fractional moments or averages of ratios of spectral dets wanted

\[
\text{mean eigv density.} = \lim_{\epsilon \to 0} \frac{\partial}{\partial z} \lim_{z \to \zeta} \frac{\partial}{\partial \zeta} \left\langle \frac{\det[\epsilon^2 I + (zI - W)(zI - W)^\dagger]}{\det[\epsilon^2 I + (\zeta I - W)(\zeta I - W)^\dagger]} \right\rangle
\]

- other classical groups?
References:


7. Friedman W A and Mello P A 1985 Marginal distribution of an arbitrary square submatrix


