Rigidity of Graphs and Frameworks

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Frameworks in $\mathbb{R}^d$

A *framework* in $\mathbb{R}^d$ is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$.

We consider the framework to be a straight line embedding of G in $\mathbb{R}^d$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance between the points $p(u)$ and $p(v)$.

We say that (G, p) is a *realisation* of G in $\mathbb{R}^d$.

Rigid Frameworks

Let (G, p) and (G, q) be frameworks. Then:

- (G, p) and (G, q) are *equivalent* if they have the same edge lengths.
- (G, p) and (G, q) are *congruent* if $|p(u) - p(v)| = |q(u) - q(v)|$ for all $u, v \in V$.
- (G, p) is *rigid* if there exists an $\varepsilon > 0$ such that every framework (G, q) which is equivalent to (G, p) and satisfies $|p(v) - q(v)| < \varepsilon$ for all $v \in V$, is congruent to (G, p). (This is equivalent to saying that there is no 'smooth deformation' of (G, p) which preserves the lengths of all its edges.)
- (G, p) is *globally rigid* if every framework (G, q) which is equivalent to (G, p), is congruent to (G, p).

The Rigidity Matrix and the Rigidity Matroid

Let $G = (V, E)$ be a graph and d be a positive integer.

We associate a d-dimensional vector of distinct variables $\mathbf{v}$ with each $v \in V$.

The d-dimensional *rigidity matrix* M_G of G is the $|E| \times d|V|$ matrix with rows indexed by E and columns indexed by the variables in $\mathbf{v}$, $v \in V$, in which the entry in row e and columns $\mathbf{v}$ is:

$$\begin{array}{cl} \mathbf{v} - w & \text{if } e = vw \text{ is incident with } v \\ \mathbf{0} & \text{if } e \text{ is not incident with } v. \end{array}$$

The *rigidity matroid* $M(G) = (E, r)$ of G is the row matroid of M_G.

Example

$$M_G = \begin{pmatrix} \mathbf{v}_1 - v_2 & \mathbf{v}_2 - v_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2 - v_3 & \mathbf{v}_3 - v_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_3 - v_4 & \mathbf{v}_4 - v_3 \\ \mathbf{v}_1 - v_4 & \mathbf{0} & \mathbf{0} & \mathbf{v}_4 - v_1 \\ \mathbf{0} & \mathbf{v}_2 - v_4 & \mathbf{0} & \mathbf{v}_4 - v_2 \end{pmatrix}$$

$d = 1$, $\mathbf{v}_i = (x_i)$.

$$M_G = \begin{pmatrix} x_1 - x_2 & x_2 - x_1 & 0 & 0 \\ 0 & x_2 - x_3 & x_3 - x_2 & 0 \\ 0 & 0 & x_3 - x_4 & x_4 - x_3 \\ x_1 - x_4 & 0 & 0 & x_4 - x_1 \\ 0 & x_2 - x_4 & 0 & x_4 - x_2 \end{pmatrix}$$

$M(G)$ is the cycle matroid of G, $r(E) = 3$.

$d = 2$, $\mathbf{v}_i = (x_i, y_i)$.

$$M_G =$$

$$\begin{pmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - \\ 0 & 0 & x_2 - x_4 & y_2 - y_4 & 0 & 0 & x_4 - x_2 & y_4 - \end{pmatrix}$$

$M(G) = U_{5,5}$, $r(E) = 5$.

The *rigidity matrix* of a framework (G, p) is the matrix $M_{G,p}$ obtained from M_G by replacing $\mathbf{v}$ by $p(v)$ for all $v \in V$.
The *rigidity matroid* of (G, p), $M(G, p)$, is the row matroid of $M_{G,p}$.

We say that (G, p) is a *generic* realization of G if $M(G, p) = M(G)$. (This occurs, in particular, when the co-ordinates of all the points $p(v)$, $v \in V$, are algebraically independent over $\mathbb{Q}$.)

Infinitesimally Rigid Frameworks

Let

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+1 \\ \binom{n}{2} & \text{if } n \leq d+1 \end{cases}$$

Lemma 1 Let (G, p) be a framework in $\mathbb{R}^d$. Then

$$\text{rank} M_{G,p} \leq \text{rank} M_G \leq S(n,d).$$

Furthermore, if $\text{rank} M_{G,p} = S(n,d)$, then (G, p) is rigid.

We say that (G, p) is *infinitesimally rigid* if $\text{rank} M_{G,p} = S(n,d)$. Thus

$$(G,p) \text{ is infinitesimally rigid} \Rightarrow (G,p) \text{ is rigid}$$

When (G, p) is a generic realisation of G:

(G, p) is rigid $\Leftrightarrow$ (G, p) is infinitesimally rigid

Rigid Graphs

A graph G is *rigid* in $\mathbb{R}^d$ if

$$\text{rank} M_G = S(n,d).$$

Lemma 2 The following statements are equivalent.

- G is rigid in $\mathbb{R}^d$.
- $r(E) = S(n,d)$ in $M(G)$.
- some framework (G,p) in $\mathbb{R}^d$ is infinitesimally rigid.
- every generic framework (G,p) in $\mathbb{R}^d$ is rigid.
- the set of all frameworks (G,p) which are rigid in $\mathbb{R}^d$ is a dense subset of $\mathbb{R}^{nd}$.

Minimally Rigid Graphs

A graph G is *minimally rigid* in $\mathbb{R}^d$ if G is rigid and $G-e$ is not rigid for all $e \in E$.

Lemma 3 The following statements are equivalent.

- G is minimally rigid in $\mathbb{R}^d$.
- G is rigid and $|E| = S(n,d)$.
- G is rigid and E is independent in $M(G)$.

Rigidity in $\mathbb{R}$

We have $S(n,1) = n-1$.

$M(G)$ is just the cycle matroid of G.

A set of edges E' is independent in $M(G)$ if and only if E' induces a forest in G.

G is rigid if and only if G is connected.

G is minimally rigid if and only if G is a tree.

(G,p) is a generic realization of G for all embeddings p (in which adjacent vertices are mapped to distinct points).

Rigidity in $\mathbb{R}^2$

We have $S(n,2) = 2n-3$ if $n \geq 2$.

For $X \subset V$ let $E(X)$ denote the set, and $i(X)$ the number, of edges of G induced by X.

Theorem 4 (Laman, 1970) Let $G = (V, E)$ be a graph. Then E is independent in $M(G)$ if and only if

$$i(X) \leq 2|X| - 3 \text{ for all } X \subseteq V \text{ with } |X| \geq 2.$$

Theorem 5 (Lovasz and Yemini, 1982) Let $G = (V,E)$ be a graph. Then

$$r(E) = \min\{\sum_{i=1}^{t}(2|X_i| - 3)\}$$

where the minimum is taken over all collections of subsets $\{X_1, X_2, \ldots, X_t\}$ of G such that $\{E(X_1), E(X_2), \ldots, E(X_t)\}$ partitions E.

A *rigid component* of G is a maximal rigid subgraph of G.

If $H_1, H_2, \ldots, H_t$ are the rigid components of G and $X_i = V(H_i)$ then

- $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq t$ and
- $r(E) = \sum_{i=1}^{t}(2|X_i| - 3)$.

A 0-*extension* of G is obtained by adding a new vertex v and joining v to two vertices of G.

Suppose $x,y,z \in V$ and $xy \in E$. A 1-*extension* of G is obtained by adding a new vertex v, joining v to x,y,z and deleting the edge xy.

Theorem 6 (Henneberg, 1911) If G is minimally rigid then any 0-extension or 1-extension of G is minimally rigid. Conversely, if H is minimally rigid and $|V(H)| \geq 3$, then H is a 0-extension or 1-extension of a minimally rigid graph G.

Corollary 7 G is minimally rigid if and only if G can be obtained from K_2 by a sequence of 0-extensions and 1-extensions.

Global Rigidity

A framework (G,p) is *algebraically independent* if the set of all co-ordinates of all points $p(v)$, $v \in V$, is algebraically independent over $\mathbb{Q}$.

Theorem 8 (Hendrickson, 1992) Suppose (G,p) is an algebraically independent realization of G in $\mathbb{R}^d$. If (G,p) is globally rigid then either G is a complete graph with at most $d+1$ vertices, or the following two conditions hold:
(a) G is $(d+1)$-connected, and
(b) $G-e$ is rigid in $\mathbb{R}^d$ for all $e \in E$.

We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G-e$ is rigid in $\mathbb{R}^d$ for all $e \in E$.

Theorem 9 (Connelly, 1989 and 2003) Suppose G is a 1-extension of H. Let (G, p) be an algebraically independent realization of G in $\mathbb{R}^2$. If $(H, p|_{V(H)})$ is globally rigid, then (G, p) is globally rigid.

Theorem 10 (Jordán and BJ, 2003) Let $G = (V, E)$ be a graph. Then G is 3-connected and redundantly rigid in $\mathbb{R}^2$ if and only if G can be obtained from K_4 by the operations of adding edges and performing 1-extensions.

Theorem 11 (Jordán and BJ, Connelly, 2003) Let (G, p) be an algebraically independent realization of G in $\mathbb{R}^2$. Then (G, p) is globally rigid if and only if either $G = K_2, K_3$, or the following two conditions hold:
(a) G is 3–connected, and
(b) G is redundantly rigid in $\mathbb{R}^2$.