Counting 2-Connected Deletion-Minors of Binary Matroids

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3 July 2011

Abstract

We introduce a new invariant for a binary matroid and use it to obtain upper bounds on the number of circuits and, more generally, the number of 2-connected deletion minors containing a fixed element of the matroid. We conjecture that this invariant can also be used to bound the modulus of the roots of the matroid's characteristic polynomial.

Key Words: Binary matroid, 2-connected minor, characteristic polynomial.

1 Introduction

The purpose of this paper is to introduce a new invariant for a binary matroid M and to use it to obtain upper bounds on the number of circuits and, more generally, the number of 2-connected deletion minors of M containing a fixed element. Our matroid invariant extends a graph invariant called 'maxmaxflow' previously introduced in [4].

Given a binary matroid M, let $\mathcal{B}$ be the set of all bases of the cocycle space of M and put

$$\Lambda(M) = \min_{B \in \mathcal{B}} \left\{ \max_{K \in B} \{|K|\} \right\} .$$

When M is the cycle matroid of a graph G, $\Lambda(M)$ is equal to the maximum number of edge-disjoint paths between any pair of distinct vertices of G,

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and hence can be computed in polynomial time using maximum flow calculations, see [4]. Similarly, when M is the cocycle matroid of G, $\Lambda(M)$ can be computed in polynomial time using Horton's algorithm [3] for finding a *shortest cycle basis* for G, i.e. a basis B for the cycle space of G such that the sum of the lengths of the cycles in B is as small as possible, see Chickering, Geiger and Heckerman [1] and Galbiati [2]. (It is not difficult to see that for an arbitrary binary matroid M, $\Lambda(M)$ is equal to the maximum length of a cocycle in a 'shortest basis' of the cocycle space of M.) We do not know if $\Lambda(M)$ can be determined in polynomial time for an arbitrary binary matroid M. However, the related problem of finding a minimum size cocircuit in M is known to be NP-hard, see [13].

Our interest in $\Lambda(M)$ was sparked initially by a result of Sokal [9] that the modulus of the roots of the chromatic polynomial of a graph G can be bounded above by a linear function of its maximum degree, $\Delta(G)$. It is an elementary fact that all of the *integer* chromatic roots of G lie in the interval $[0, \Delta(G)]$, i.e. the chromatic number $\chi(G)$ is at most $\Delta(G) + 1$. Sokal [9, Corollary 6.4] showed that *all* the chromatic roots (real or complex) lie in the disc $|q| < 7.963907\,\Delta(G)$. Furthermore, he conjectured, following a suggestion of Shrock and Tsai [7, 8], that it might be possible to bound all the chromatic roots in terms of the maxmaxflow $\Lambda(G)$, taking $\Lambda(G) := \Lambda(M)$ where M is the cycle matroid of G.

It is not difficult to see that we always have $\Lambda(G) \leq \Delta(G)$. This follows from the fact that the stars centred on all but one of the vertices of G span the cocycle space of G (and form a basis whenever G is connected). Thus Sokal's conjecture would generalise his result that the chromatic roots of a graph can be bounded by a function of its maximum degree. Some evidence in support of the conjecture follows from the fact that $\chi(G) \leq \Lambda(G) + 1$, see [4]. Further evidence follows from a recent result of Royle and Sokal [5] which verifies the conjecture for series parallel graphs.

An important step in Sokal's proof that the chromatic roots of G can be bounded by a function of $\Delta = \Delta(G)$ is showing that the number of connected m-edge subgraphs containing a fixed vertex of G is at most $(c\Delta)^m$ for some constant c. Our work here on counting connected deletion minors of a binary matroid M in terms of $\Lambda = \Lambda(M)$ is motivated in part by the goal of adapting the methods of [9] to bound the roots of the characteristic polynomial of M by a function of Λ (and hence verify Sokal's conjecture which corresponds to the special case when M is graphic). Our main result implies that the number of m-element circuits of M containing a fixed element e is bounded above by Λ^m. We also show that the number of 2-connected m-element deletion minors of M containing e is at most $2^{m^2/2}\Lambda^m$, and conjecture that this bound can be reduced to $(c\Lambda)^m$ for some constant c.

2 Definitions and preliminary results

We refer the reader to [6] for the basic definitions in matroid theory. Given a matroid M, we use E_M to denote the ground set of M, $\mathcal{C}_M$ the set of circuits of M, $\mathcal{K}_M$ the set of cocircuits of M, and r_M the rank function of M, suppressing the subscript M whenever it is obvious which matroid we are referring to. We denote the rank of M by $r(M)$.

For $S \subseteq E$, the matroid obtained from M by *deleting* S is the matroid $M\backslash S$ with $E_{M\backslash S} = E\backslash S$ and $\mathcal{C}_{M\backslash S} = \{C \in \mathcal{C}\colon\ C \subseteq E_M\backslash S\}$. The matroid obtained by *contracting* S is the matroid M/S with $E_{M/S} = E\backslash S$ and $\mathcal{C}_{M/S}$ equal to the set of minimal non-empty members of $\{C\backslash S\colon\ C \in \mathcal{C}\}$. Note that if e is a loop or a coloop of M then $M\backslash e = M/e$. We call a matroid N a *deletion minor* of M if $N = M\backslash S$ for some $S \subset E$, a *contraction minor* of M if $N = M/T$ for some $T \subset E$, and a *minor* of M if $N = (M\backslash S)/T$ for some disjoint subsets $S, T \subset E$. (Thus, if M is the circuit matroid of a graph G and $N = M\backslash S$, then N is the circuit matroid of the subgraph $H = G - S$ of G. Similarly, if $N = M/S$, then N is the circuit matroid of the contraction $H = G/S$ of G.)

The matroid M is *2-connected* if $E| \geq 2$ and every pair of elements of M are contained in a common circuit. (Thus, if M is the circuit or cocircuit matroid of a graph G containing at least three vertices, then M is 2-connected if and only if G is 2-connected and loopless.) We will need the following lemma of Tutte [12] (see also [6, Theorem 4.3.1]).

Lemma 2.1 *Suppose M is a 2-connected matroid with at least three elements and $e \in E$. Then at least one of the matroids $M\backslash e$ and M/e is 2-connected.*

Binary matroids

A matroid M is *binary* if there exists a vector space V over GF(2) and a map $f\colon\ E \to V$ such that, for each $S \subseteq E$, $r(S)$ is equal to the dimension of the subspace of V spanned by $f(S)$. All matroids considered henceforth are binary. (Recall, in particular, that all graphic and cographic matroids are binary.) Given a binary matroid M, we consider the set 2^E of all subsets of E as a vector space over GF(2), where vector addition is given by symmetric difference (which we denote by $\oplus$). The *cycle space* and *cocycle space* of M are the subspaces of 2^E spanned by $\mathcal{C}$ and $\mathcal{K}$, respectively. The dimension of the cycle space is $|E| - r(E)$ and the dimension of the cocycle space is $r(E)$. We refer to the elements of the cycle and cocycle spaces as *cycles* and *cocycles* of M. We will need the following three elementary lemmas for binary matroids.

Lemma 2.2 *[6, Proposition 9.2.2] Let M be a binary matroid and $S \subseteq E$. Then S is a cycle (respectively cocycle) of M if and only if $|S \cap T|$ is even for all cocycles (respectively cycles) T of M.*

Lemma 2.3 *Let M be a binary matroid and e be an element of M which is not a loop. Let B be a basis for the cocycle space of M and choose $K_1 \in B$ with $e \in K_1$. Let $X = \{K \in B\colon e \in K\}$ and $Y = B \backslash X$. Then*
(a) $B_1 = \{K \oplus K_1\colon K \in X \backslash \{K_1\}\} \cup Y$ is a basis for the cocycle space of M/e.
(b) If e is not a coloop of M, then $B_2 = \{K - \{e\}\colon K \in X\} \cup Y$ is a basis for the cocycle space of $M \backslash e$.

Proof: (a) Each element of B_1 is a cocycle of M which does not contain e, and hence is a cocyle of M/e. Since e is not a loop of M we have $r(M/e) = r(M) - 1$. Hence the dimension of the cocycle space of M/e is one less than the dimension of the cocycle space of M and it suffices to show that B_1 is linearly independent. Suppose $[\bigoplus_{K \in X'} (K \oplus K_1)] \oplus [\bigoplus_{K \in Y'} K] = \emptyset$, for some $X' \subseteq X - \{K_1\}$ and $Y' \subseteq Y$. Then either $[\bigoplus_{K \in X' \cup Y'} K] = \emptyset$ or $[\bigoplus_{K \in X' \cup Y'} K] \oplus K_1 = \emptyset$. Both alternatives contradict the linear independence of B.

(b) Each element of B_2 is a cocycle of $M \backslash e$. Since e is not a coloop of M we have $r(M \backslash e) = r(M)$. Hence the cocycle spaces of M and $M \backslash e$ have the same dimension and it suffices to show that B_2 is linearly independent. Suppose $[\bigoplus_{K \in X'} (K - \{e\})] \oplus [\bigoplus_{K \in Y'} K] = \emptyset$, for some $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' \neq \emptyset$. Then $\bigoplus_{K \in X' \cup Y'} K \subseteq \{e\}$. This is impossible: the left hand side of the above set inclusion cannot be empty since B is linearly independent, and cannot equal $\{e\}$ since it belongs to the cocycle space of M, and $\{e\}$ is not a cocycle of M (because it is not a coloop). •

Lemma 2.4 *Let M be a binary matroid and e be an element of M. Let $B = \{K_1, K_2, \ldots, K_m\}$ be a basis for the cocycle space of $M \backslash e$.*
(a) If e is a coloop of M, then $B_1 = B \cup \{\{e\}\}$ is a basis for the cocycle space of M.
(b)If e is not a coloop of M, then there exists a basis $B_2 = \{K_1', K_2', \ldots, K_m'\}$ for the cocycle space of M such that $K_i \subseteq K_i' \subseteq K_i \cup \{e\}$ for all i, $1 \leq i \leq m$.

Proof: (a) Since e is a coloop of M, $\{e\}$ is a cocycle of M, and $r(M) = r(M\backslash e) + 1$. Hence the dimension of the cocycle space of M is $m+1$. It follows from the definition of $M\backslash e$ that either K_i or $K_i \cup \{e\}$ is a cocycle of M for all i, $1 \leq i \leq m$. However, if $K_i \cup \{e\}$ is a cocycle of M, then $(K_i \cup \{e\}) \oplus \{e\} = K_i$ is also a cocycle of M. Hence K_i is a cocycle of M for all i, $1 \leq i \leq m$. The linear independence of B_1 follows from the linear independence of B and the fact that $e \notin K_i$ for all i, $1 \leq i \leq m$.

(b) Since e is a not a coloop of M, $r(M) = r(M\backslash e)$, and hence the dimension of the cocycle space of M is m. It follows from the definition of $M\backslash e$ that either K_i or $K_i \cup \{e\}$ is a cocycle of M for all i, $1 \leq i \leq m$. Let K_i' be the cocycle of M with $K_i \subseteq K_i' \subseteq K_i \cup \{e\}$ and put $B_2 = \{K_1', K_2', \ldots, K_m'\}$. The linear independence of B_2 follows from the linear independence of B and the fact that $\{e\}$ is not a cocycle of M. •

Corollary 2.5 *Let M be a binary matroid, $S \subseteq E$ and $B = \{K_1, K_2, \ldots, K_m\}$ be a basis for the cocycle space of $M\backslash S$. Then there exists a basis $B' = \{K_1', K_2', \ldots, K_n'\}$ for the cocycle space of M such that $K_i \subseteq K_i' \subseteq K_i \cup S$ for all i, $1 \leq i \leq m$, and $K_i' \subseteq S$ for all i, $m+1 \leq i \leq n$.*

Proof: This follows from Lemma 2.4 by induction on $|S|$. •

Weighted binary matroids

A *weighted binary matroid* is a pair $(M, \mathbf{w})$ where M is a binary matroid and $\mathbf{w} = \{w_e\}_{e \in E}$ is a set of nonnegative real weights assigned to the elements of M. Let $\mathcal{B}$ be the set of all bases of the cocycle space of M and put

$$\Lambda(M, \mathbf{w}) \;=\; \min_{B \in \mathcal{B}} \left\{ \max_{K \in B} \left\{ \sum_{e \in K} w_e \right\} \right\} .$$

Thus the invariant $\Lambda(M)$ defined in Section 1 can be obtained from $\Lambda(M, \mathbf{w})$ by taking all weights equal to one. We consider the weighted version $\Lambda(M, \mathbf{w})$ for two reasons: the first is that our proofs for the weighted and unweighted versions are identical; the second is that we believe $\Lambda(M, \mathbf{w})$ can be used to bound not only the roots of the univariate characteristic polynomial of M (by taking $w_e = 1$ for all $e \in E$), but also the bivariate Tutte polynomial of M (by taking $w_e = w$ for all $e \in E$), and the multivariate Potts model partition function (by taking arbitrary edge weights), see [11].

Given a weighted binary matroid $(M, \mathbf{w})$ and $e \in E$, Lemma 2.3 immediately gives the following upper bounds for $\Lambda(M\backslash e, \mathbf{w})$ and $\Lambda(M/e, \mathbf{w})$ in terms of $\Lambda(M, \mathbf{w})$. (Here and henceforth we abuse notation by using $\mathbf{w}$ for both the weight function M and its restriction to a minor of M.)

Lemma 2.6 *Let $(M, \mathbf{w})$ be a weighted binary matroid and e be an element of M that is neither a loop nor a coloop. Then $\Lambda(M/e, \mathbf{w}) \leq 2\Lambda(M, \mathbf{w})$ and $\Lambda(M\backslash e, \mathbf{w}) \leq \Lambda(M, \mathbf{w})$.*

3 Main results

Given a weighted binary matroid $(M, \mathbf{w})$ and $T \subseteq E$ we define the *weight of* T to be $w_T = \prod_{e \in T} w_e$. We first obtain bounds on the sum of the weights of the m-element circuits of M which contain a fixed set of elements. For $S \subseteq E$ let

$$\mathcal{C}_m(S) = \{C \in \mathcal{C} \ : \ |C| = m \text{ and } S \subseteq C\}$$

and put $c_m(S) = \sum_{C \in \mathcal{C}_m(S)} w_C$.

Theorem 3.1 *Let $(M, \mathbf{w})$ be a weighted binary matroid and $S \subseteq E$ with $|S| = s \geq 1$. Let B be a basis for the cocycle space of $M\backslash S$ and put $\Omega = \max_{K \in B} \left\{\sum_{e \in K} w_e\right\}$. Then*

$$\sum_{m \geq 1} \Omega^{-m+s} c_m(S) \ \leq \ w_S.$$

Proof. We will show that

$$\sum_{m=1}^{k} \Omega^{-m+s} c_m(S) \ \leq \ w_S \tag{1}$$

for all $k \geq 1$. If $k < s$ then $c_m(S) = 0$ for all $1 \leq m \leq k$ and (1) holds trivially. Hence we may suppose that $k \geq s$. We proceed by induction on $k - s$. If $k = s$ then $c_m(S) = 0$ for all $1 \leq m < k$, $c_k(S) \leq w_S$ and again (1) holds. Hence suppose that $k > s$. Let B' be a basis for the cocycle space of M obtained from B as in Corollary 2.5. Then

$$\sum_{e \in K-S} w_e \leq \Omega \text{ for all } K \in B'. \tag{2}$$

Suppose $|S \cap K|$ is even for all $K \in B'$. Then $|S \cap K|$ is even for all cocycles K of M and hence S is a cycle of M by Lemma 2.2. Thus $c_m(S) = 0$

if either $m \neq s$ or S is not a circuit of M, and $c_s(S) = w_S$ if S is a circuit of M. Thus (1) holds.

Hence we may assume that $|S \cap K|$ is odd for some $K \in B'$. Let $K = \{e_1, e_2, \ldots, e_n\}$. Then $|C \cap K|$ is even for all $C \in \mathcal{C}_m(S)$ by Lemma 2.2. Since $|S \cap K|$ is odd, it follows that $C \cap K \not\subseteq S$. We shall classify the circuits $C \in \mathcal{C}_m(S)$ according to $p(C) = \min\{i : e_i \in (C \cap K) - S\}$. Let $\mathcal{C}^i = \{C \in \mathcal{C}_m(S) \ : \ p(C) = i\}$. Note that $\mathcal{C}^i \subseteq \mathcal{C}_m(S \cup \{e_i\})$ for all $1 \leq i \leq n$. Using induction we deduce that:

$$\begin{aligned}
\sum_{m=1}^{k} \Omega^{-m+s} c_m(S) &= \sum_{m=1}^{k} \Omega^{-m+s} \sum_{i=1}^{n} \sum_{C \in \mathcal{C}^i} w_C \\
&\leq \sum_{m=1}^{k} \Omega^{-m+s} \sum_{e_i \in K-S} c_m(S \cup \{e_i\}) \\
&= \Omega^{-1} \sum_{e_i \in K-S} \sum_{m=1}^{k} \Omega^{-m+s+1} c_m(S \cup \{e_i\}) \\
&\leq \Omega^{-1} \sum_{e_i \in K-S} \mathbf{w}(S \cup \{e_i\}) \\
&= \Omega^{-1} w_S \sum_{e_i \in K-S} w_{e_i} \\
&\leq w_S
\end{aligned}$$

by (2). •

Theorem 3.1 has the following immediate corollary.

Corollary 3.2 *Let $(M, \mathbf{w})$ be a weighted binary matroid and $S \subseteq E$ with $|S| = s \geq 1$. Let $\Lambda(M \backslash S, \mathbf{w}) = \Lambda$. Then $c_m(S) \leq w_S \Lambda^{m-s}$ for all $m \geq 1$.*

The special cases when $|S| = 1$ in Theorem 3.1 and Corollary 3.2 are closely related to results for graphic matroids given in [4, Proposition 5.3, Corollary 5.4]. An example in the same paper [4, Examples 5.4] shows that the exponential growth rate $c_m(e) \sim \Lambda^{m-1}$ cannot be improved even for the special case of graphic matroids.

Given a weighted binary matroid (M, w) and N a minor of M the *weight of* N is $w_N = \prod_{e \in E_N} w_e$. We next obtain a bound on the sum of the weights of the 2-connected m-element deletion minors of M which contain a particular element. For $e \in E$, let $\mathcal{D}_m(e, M)$ denote the set of all 2-connected m-element deletion minors of M which contain e, and put $d_m(e, M) = \sum_{N \in \mathcal{D}_m(e,M)} w_N$.

Theorem 3.3 *Let* $(M, \mathbf{w})$ *be a weighted binary matroid and* $e \in E$. *Then*

$$d_m(e, M) \leq \tfrac{1}{2} w_e \, \Lambda(M, \mathbf{w})^{m-1} \prod_{i=0}^{m-2} (1 + 2^i)$$

for all $m \geq 2$.

Proof: We use induction on m. If e is a loop or coloop of M, then $d_m(e, M) = 0$ for all $m \geq 2$. Hence we may suppose that e is not a loop or coloop of M. Let B be a basis for the cocycle space of M such that $\sum_{f \in K} w_f \leq \Lambda(M, \mathbf{w})$ for all $K \in B$. Choose $K \in B$ with $e \in K$ and let $K = \{e, e_1, \ldots, e_t\}$.

Suppose $m = 2$ and let $F = \{f \in E\colon \{e, f\} \in \mathcal{C}\}$. Then $d_2(e, M) = w_e \sum_{f \in F} w_f$. Since F is a subset of each cocycle of M which contains e, we have $\sum_{f \in F} w_f \leq \sum_{f \in K} w_f \leq \Lambda(M, \mathbf{w})$. Thus the theorem holds for $m = 2$ and we may assume that $m \geq 3$.

For each 2-connected deletion minor N of M with $e \in E_N$, we have $|E_N \cap K| \geq 2$ (since, if C is a circuit of N containing e, then C is a circuit of M and hence $|K \cap C| \neq 1$ by Lemma 2.2). We classify the deletion minors $N \in \mathcal{D}(e, M)$ according to $p(N) := \min\{i\colon e_i \in E_N,\ 1 \leq i \leq t\}$. Let $\mathcal{D}^i = \{N \in \mathcal{D}_m(e, M) : p(N) = i\}$. Using Lemma 2.1, we may deduce that if $N \in \mathcal{D}^i$, then either $N \backslash e_i \in \mathcal{D}_{m-1}(e, M \backslash e_i)$ or $N / e_i \in \mathcal{D}_{m-1}(e, M)$ or both. Thus

$$d_m(e, M) \ \leq \ \sum_{i=1}^{t} w_{e_i} [d_{m-1}(e, M \backslash e_i) + d_{m-1}(e, M / e_i)] \, .$$

The theorem now follows by applying Lemma 2.6 and induction, using the fact that $\sum_{e_i \in K} w_{e_i} \leq \Lambda(M, \mathbf{w})$. •

Theorem 3.3 implies that $d_m(e, M) \leq 2^{m^2/2} \Lambda(M, \mathbf{w})^m$. We conjecture that a stronger result is true.

Conjecture 3.4 *There exists a universal constant* c *such that if* $(M, \mathbf{w})$ *is a weighted binary matroid and* $e \in E$, *then* $d_m(e, M) \leq (c \Lambda(M, \mathbf{w}))^m$.

This conjecture is true for graphic matroids, with $c = 2/\ln 2$, by [4, Corollary 8.5], but even the unweighted cographic case is still open.

Conjecture 3.5 *There exists a universal constant c such that if* $G = (V, E)$ *is a graph,* $e \in E$ *and* B *is a basis for the cycle space of* G *with* $|C| \leq \Omega$ *for all* $C \in B$*, then the number of* m*-edge 2-connected contractions of* G *is at most* $(c\,\Omega)^m$*.*

The reader may wonder why a similar argument to that given in the proof of Theorem 3.1 cannot be used to verify Conjecture 3.4 or 3.5. The problem is that we cannot use a parity argument to deduce that there exists a $K \in B'$ with the property that $E_N \cap K \not\subseteq S$ for all 2-connected deletion minors N of M with $S \subseteq E_N$.

Acknowledgments

I would like to thank Alan Sokal for valuable conversations.

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