A Note on the Erdös-Farber-Lovász Conjecture

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Abstract

A hypergraph H is linear if no two distinct edges of H intersect in more than one vertex and loopless if no edge has size one. A q-edge-colouring of H is a colouring of the edges of H with q colours such that intersecting edges receive different colours. We use Δ_H to denote the maximum degree of H. A well known conjecture of Erdös, Farber and Lovász is equivalent to the statement that every loopless linear hypergraph on n vertices can be n-edge-coloured. In this note we show that the conjecture is true when the partial hypergraph S of H determined by the edges of size at least three can be Δ_S-edge-coloured and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$.

1 Introduction and terminology

A hypergraph H on a finite set $V(H) = \{v_1, v_2, \dots v_n\}$ is a family $E(H) = \{e_1, e_2, \dots, e_m\}$ of non-empty subsets of $V(H)$ such that $\bigcup_{i=1}^{m} e_i = V(H)$. The elements of $V(H)$ are called the *vertices* and the elements of $E(H)$ the *edges* of H. A hypergraph can also be defined by its incidence matrix $A(H) = [a_{ij}]$, with rows representing the vertices $v_1, \dots, v_n$, columns representing the edges $e_1, e_2, \dots, e_m$, where $a_{ij} = 1$ when $v_i \in e_j$ and 0 otherwise. The *dual* H^* of H is the hypergraph with vertex set

$E(H)$, edge set $V(H)$ and incidence matrix $A(H)^T$. A hypergraph in which each edge has size at most two is a graph (without isolated vertices).

A *subhypergraph* of H is a hypergraph which corresponds to a submatrix of the incidence matrix $A(H)$. A *partial hypergraph* of H is a subhypergraph H' with $E' = E(H') \subseteq E(H)$ and $V(H') = \cup_{e \in E'} e$. We shall say that H' is *determined by* E'. The columns of $A(H')$ are just the columns of $A(H)$ corresponding to the edges in E'. We denote the partial hypergraph determined by $E(H) \setminus E'$ by $H - E'$ (or by $H - e$ in the case where $E' = \{e\}$).

A hypergraph H is said to be *linear* if $|e \cap f| \leq 1$ for all $e, f \in E(H)$. An edge of size one is called a *loop* and a hypergraph in which each edge has size at least two is called *loopless*. A loopless linear graph is said to be *simple*.

For $v \in V(H)$, $d_H(v)$ is the number of edges containing v in H and $\Delta_H = \max_{v \in V(H)} d_H(v)$. The maximum number of pairwise intersecting edges of H is denoted by Δ_H^0.

A k-vertex colouring of H is an assignment of k colours to the vertices of H in such a way that no edge contains two vertices of the same colour. Similarly, a k-edge colouring of H is an assignment of k colours to the edges of H so that distinct intersecting edges receive different colours. The chromatic index $q(H)$ is the least number k of colours required for a k-edge colouring of H. Clearly

$$q(H) \geq \Delta_H^0 \geq \Delta_H.$$

A hypergraph H is said to have the *edge-colouring property* if $q(H) = \Delta_H$.

This note was motivated by the following well-known conjecture due to Erdös, Farber and Lovász (see [4]).

Conjecture 1 *Let H be a linear hypergraph consisting of n edges, each of size n. Then it is possible to colour the vertices of H with n colours so that no two vertices in the same edge receive the same colour.*

Let H be a linear hypergraph and let $V' \subseteq V(H)$ be the set of vertices occurring in at least two edges of H. If it is possible to colour the vertices in V' so that no two vertices in the same edge receive the same colour, then this colouring can be extended to a vertex colouring of H with the same number of colours. Furthermore, if H has n edges, then since H is linear, no edge can contain more than $n-1$ vertices of degree at least two. Thus Conjecture 1 is equivalent to the following:

Conjecture 2 *Let H be a linear hypergraph consisting of n edges, in which every vertex has degree at least 2. Then it is possible to colour the vertices of H with n colours so that no two vertices in the same edge receive the same colour.*

It is easily seen that the dual of a linear hypergraph is also linear. Further, the dual of the condition that no vertex has degree less than 2 is that no edge should contain less than two vertices. Hence Conjecture 2 is equivalent to the following:

Conjecture 3 *Let H be a loopless linear hypergraph on n vertices. Then $q(H) \leq n$.*

Let S be the partial hypergraph of H determined by the edges of size at least three. Conjecture 3 is true if $S = \emptyset$, since every simple graph on n vertices can be n-edge coloured. We shall show Conjecture 3 is true for all H for which S has the edge-colouring property and $\Delta_S \leq 3$.

2 Results

Throughout this section, H denotes a loopless linear hypergraph on n vertices. We use the following notation. The partial hypergraph of H determined by the edges of size at least 3 is denoted by S. Note that every edge in the partial hypergraph $G = H - E(S)$ has size 2 and hence G is a simple graph. We denote the subgraph of G induced by the set of vertices in $V(H) \setminus V(S)$ by T and the subgraph induced by the vertices of degree Δ_G by G_Δ. Our general approach to edge-colouring H is to extend a $q(S)$-edge-colouring of S to a subset $E' \subseteq E(H) \setminus E(S)$, so that the partial hypergraph $G' = H - (E(S) \cup E')$ can be edge-coloured with the remaining $n - q(S)$ colours. To edge-colour G', we use the following well-known theorems due to Vizing [6] and Fournier [5].

Theorem 2.1 (Vizing) *Let G be a simple graph. Then $q(G) \leq \Delta_G + 1$.*

Theorem 2.2 (Fournier) *Let G be a simple graph. If G_Δ is acyclic, then $q(G) = \Delta_G$.*

Our first lemma follows from a stronger theorem of Berge and Hilton (Theorem C in [2]). We include their proof for completeness.

Lemma 2.3 *Let H be a loopless linear hypergraph on n vertices. If $\Delta_S = 1$, then $q(H) \leq \Delta_H + 1$.*

Proof. Give the edges of S the colour c. Choose a maximum matching M in $H - V(S)$ and give the edges of M the colour c also. Then the partial hypergraph $G = H - E(S) \cup M$ is a simple graph in which either $\Delta_G = \Delta_H - 1$, or $\Delta_G = \Delta_H$ and the vertices of degree Δ_H are independent. Hence by Theorem 2.1 or Theorem 2.2, G is Δ_H-edge-colourable, giving $q(H) \leq \Delta_H + 1$. □

Since H is loopless and linear, $\Delta_H \leq n-1$ and so Lemma 2.3 implies Conjecture 3 is true when $\Delta_S = 1$. In the case when $\Delta_S \geq 2$, we make the stronger assumption that H has the edge-colouring property. We can also assume, without loss of generality, that every pair of vertices occur together in an edge, since adding edges of size two cannot decrease $q(H)$. This has the following simple consequences. When $T \neq \emptyset$, T is a complete graph and each vertex of T is joined by an edge of size two to each vertex of S. If $x, y \in V(S)$, then x and y are non-adjacent in the graph $G = H - E(S)$ if and only if in H they are contained in the same edge $e \in E(S)$. In particular, for each $v \in V(S)$, $d_G(v) \leq n-3$ and $d_G(v) = n-3$ if and only if v is contained in a unique edge $e \in E(S)$ with $|e| = 3$. Further, when $d_S(v) = 2$ we have $d_G(v) \leq n-5$, and when $d_S(v) = 3$ we have $d_G(v) \leq n-7$.

Lemma 2.4 *Let H be a loopless linear hypergraph on n vertices in which $\Delta_S = q(S) = 2$. Then $q(H) \leq n$.*

Proof. We may assume, without loss of generality, that each pair of vertices is contained in an edge. Give S a 2-edge-colouring with colours c_1, c_2. If $V(T) = \emptyset$, let $G = H - E(S)$. Then $\Delta_G \leq n-3$ and G is $(n-2)$-edge-colourable, by Vizing's theorem, giving $q(H) \leq n$.

Now suppose $V(T) = \{u\}$. If there exists a vertex $w \in V(S)$ such that $d_S(w) = 1$, then there is a colour, say c_1, missing at w. Give uw the colour c_1 and let $G = H - E(S) - uw$. Then $V(G_\Delta) = \{u\}$ and $\Delta_G = n-2$. Hence G can be $(n-2)$-edge-coloured by Fournier's theorem and, again, $q(H) \leq n$. Otherwise, every vertex in S is incident with an edge in each colour and hence $d_G(v) \leq n-5$ for all $v \in V(S)$. Choose an edge $e \in E(S)$ such that e is in colour c_2 and change the colour on e to a new colour c_3. Then S contains vertices w_2, w_3 such that colour c_j is missing at w_j, for each $j \in \{2, 3\}$. Give the edge uw_j the colour c_j, for $j = 2, 3$, and let $G = H - E(S) \cup \{uw_2, uw_3\}$. Then $V(G_\Delta) = \{u\}$ and $\Delta_G = n-3$. Thus G is $(n-3)$-edge-colourable, by Fournier's theorem, and again $q(H) \leq n$.

Next, suppose $V(T) = \{u_1, u_2\}$. Give u_1u_2 the colour c_1 and let $G = H - E(S) - u_1u_2$. Then $V(G_\Delta) = \{u_1, u_2\}$, $\Delta_G = n-2$ and G is $(n-2)$-edge-colourable by Fournier's theorem, giving $q(H) \leq n$.

Finally suppose $|V(T)| = t \geq 3$. If t is even, choose two disjoint perfect matchings M_1, M_2 in T and colour the edges of M_i with colour c_i, for each $i \in \{1, 2\}$. Let $G = H - E(S) \cup M_1 \cup M_2$. Then $\Delta_G = n-3$ and we can $(n-2)$-edge-colour G, by Vizing's theorem. If t is odd, let u_1, u_2 be distinct vertices in T. Let M_i be a perfect matching in $T - u_i$ and colour the edges of M_i with colour c_i, for $i = 1, 2$. Let $G = H - E(S) \cup M_1 \cup M_2$. Then $V(G_\Delta) = \{u_1, u_2\}$ and $\Delta_G = n-2$. Hence G is again $(n-2)$-edge-colourable, by Fournier's theorem. Thus, in both cases, H is n-edge-colourable. $\square$

Lemma 2.5 *Let H be a loopless linear hypergraph on n vertices in which $\Delta_S = q(S) = 3$. Then $q(H) \leq n$.*

Proof. We may assume, without loss of generality, that each pair of vertices is contained in an edge. Give S a 3-edge-colouring with colours c_1, c_2, c_3. Let $G_0 = H - E(S)$ and let $X = \{x \in V(S) : d_{G_0}(x) = n-3\}$. Then if $v \in X$, v is incident with just one edge in S, and this edge has size 3. Let $X_i \subseteq X$ be the subset of vertices in X that are incident with an edge in colour c_i, $i = 1,2,3$. Number the colours so that $|X_3| \leq |X_2| \leq |X_1|$. If $X_3 \neq \emptyset$, construct a matching M_{32} in G_0 of X_3 into X_2, saturating the vertices of X_3. Similarly, if $X_2 \neq \emptyset$, construct a matching M_{21} in G_0 of X_2 into X_1, saturating X_2. If any vertex of X_1 is unsaturated by M_{21}, construct a maximum matching M_{11} in G_0 between the M_{21}-unsaturated vertices in X_1. Give the edges in M_{32} the colour c_1 and the edges of M_{21} and M_{11} the colour c_3. Let $Y \subseteq X_1$ be the subset of vertices that are unsaturated by both M_{21} and M_{11}. Note that if $Y \neq \emptyset$, then Y is a subset of the vertices of a unique edge $e \in E(S)$, where $|e| = 3$, and hence Y is an independent set with $|Y| \leq 3$. Let $G_1 = G_0 - M_{32} \cup M_{21} \cup M_{11}$. We distinguish four cases.

Case (a) $V(T) = \emptyset$. Then if $Y = \emptyset$, $\Delta_{G_1} \leq n-4$ and G_1 is $(n-3)$-edge-colourable by Vizing's theorem. Otherwise, $V(G_{1_\Delta}) = Y$ and hence G_1 is again $(n-3)$-edge-colourable, by Fournier's theorem, giving $q(H) \leq n$.

Case (b) $|T| = t \geq 3$. When t is even, choose three pairwise disjoint perfect matchings, M_1, M_2, M_3, in T and give the edges of M_i the colour c_i, for $i = 1,2,3$. Let $G = G_1 - M_1 \cup M_2 \cup M_3$. Then $d_G(v) = n-4$, for all $v \in V(T)$, and hence G is $(n-3)$-edge-colourable by a similar argument to case (a). When t is odd, choose a vertex $u_3 \in V(T)$ and a perfect matching M_3 in $T - u_3$. Choose $u_1, u_2 \in V(T)$ such that $u_1u_2 \in M_3$. Choose disjoint perfect matchings M_1, M_2 in $T - M_3 - u_1$ and $T - M_3 - u_2$ respectively, and give the edges of M_i the colour c_i, for $i = 1,2,3$. If $Y = \emptyset$, let $G = G_1 - M_1 \cup M_2 \cup M_3$. Then $\Delta_G = n-3$, G_Δ is the path $u_1u_3u_2$ and hence G is $(n-3)$-edge-colourable, by Fournier's theorem. Otherwise, let $y \in Y$. Give yu_j the colour c_j, for each $j \in \{2,3\}$, and let $G = G_1 - M_1 \cup M_2 \cup M_3 \cup \{yu_2, yu_3\}$. Then $\Delta_G = n-3$ and $V(G_\Delta) = \{u_1\} \cup (Y \setminus \{y\})$. Thus G_Δ is acyclic and hence G is $(n-3)$-edge-colourable, by Fournier's theorem. This implies in each case that $q(H) \leq n$.

Case (c) $V(T) = \{u_1, u_2\}$. Suppose first $Y \neq \emptyset$. Give u_1u_2 the colour c_1. If $Y = \{y_1, y_2, y_3\}$ or $Y = \{y_1, y_2\}$, give y_1u_1, y_2u_2 the colours c_2 and c_3 respectively, and let $G = G_1 - \{u_1u_2, y_1u_1, y_2u_2\}$. Then $\Delta_G = n-3$ and G_Δ is the path $u_1y_3u_2$ in the first case and the isolated vertices u_1, u_2 in the second case. When $Y = \{y\}$, give yu_1 colour c_2 and yu_2 the colour c_3, and let $G = G_1 - \{u_1u_2, yu_1, yu_2\}$. Then G_Δ is again the independent vertices u_1, u_2 and $\Delta_G = n-3$. Hence in all three cases G is

$(n-3)$-edge-colourable, by Fournier's theorem, so that $q(H) \leq n$. Thus we may assume that $Y = \emptyset$ and hence $d_{G_1}(v) \leq n-4$, for all $v \in V(S)$.

If there exists a vertex $v \in V(S)$ which is incident with only one colour in $S_0 = S \cup M_{32} \cup M_{21} \cup M_{11}$, say with c_3, then give vu_i the colour c_i, for $i = 1,2$, give u_1u_2 the colour c_3 and let $G = G_1 - \{u_1u_2, vu_1, vu_2\}$. If no such vertex exists, but there are two distinct vertices $x, y \in V(S)$ such that c_i is missing at x and c_j is missing at y in S_0, where $i, j \in \{1,2,3\}$ and possibly $i = j$, give xu_1 the colour c_i, yu_2 the colour c_j, and give u_1u_2 a colour c_k such that $k \in \{1,2,3\} \setminus \{i, j\}$. Let $G = G_1 - \{u_1u_2, xu_1, yu_2\}$. Then in both these cases, $V(G_\Delta) = \{u_1, u_2\}$ and $\Delta_G = n-3$, so that G is $(n-3)$-edge-colourable by Fournier's theorem and $q(H) \leq n$.

Thus we may assume that every vertex in $V(S)$ is incident with edges in at least two colours in S_0, and there is at most one vertex incident with edges in only two colours. This implies that $X = \emptyset$ so that $S_0 = S$ and $\Delta_{G_1} \leq n-5$. Let $e \in E(S)$ be an edge in colour c_1 and let $f \in E(S) \setminus \{e\}$. Change the colour on e to a new colour c_4. Then since H is linear, we can find distinct vertices $w_1, w_2 \in e \setminus f$ and $z_1, z_2 \in f \setminus e$. For $i = 1,2$, give u_iw_i the colour c_1 and u_iz_i the colour c_4. Give u_1u_2 the colour c_2 and let $G = G_0 - \{u_1u_2, u_1w_1, u_2w_2, u_1z_1, u_2z_2\}$. Then $V(G_\Delta) = \{u_1, u_2\}$, $\Delta_G = n-4$ and G is $(n-4)$-edge-colourable, by Fournier's theorem, giving $q(H) \leq n$.

Case (d) $V(T) = \{u\}$. We consider the following five subcases.

Subcase (i) There exist two vertices $x, y \in V(S)$ such that there are distinct colours, c_1 and c_2 say, missing at x and y, respectively, in S_0. We extend the 3-edge-colouring of S_0 by colouring the edge ux and uy with the colours c_1 and c_2, respectively. Let $G = G_1 - \{ux, uy\}$. Then $\Delta_G = n-3$ and $V(G_\Delta) \subseteq \{u\} \cup Y$. Thus G_Δ is a star, centre u, and so $q(G) = n-3$ by Fournier's Theorem. Hence $q(G) \leq n$. Henceforth we may assume that this subcase does not occur.

Let $Z_i = \{z \in V(S) : d_{S_0}(v) = i\}$ for $1 \leq i \leq 3$. Since subcase (i) does not occur, $|Z_1| \leq 1$ and, if equality occurs, then $Z_2 = \emptyset$.

Subcase (ii) $|Z_1| = 1$. Let $Z_1 = \{z\}$. Then $d_{G_1}(v) \leq n-5$ for all $v \in S - z$. Let e_1 be the edge of S containing z and suppose $\{x, y, z\} \subseteq e_1$. Without loss of generality, we may suppose that e_1 is coloured c_1. Let e_2 be the edge of $S_0 - e_1$ coloured c_2 containing x. We modify the 3-edge-colouring of S_0 by recolouring the edge e_2 with a new colour c_4 and then extend this 4-edge-colouring by colouring the edges ux, uz, uy with the colours c_2, c_3, c_4, respectively. Let $G = G_1 - \{ux, uz, uy\}$. Then $\Delta_G = n-4$ and $G_\Delta \subseteq \{u, z\}$. Thus $q(G) = n-4$ by Fournier's Theorem and hence $q(G) \leq n$. Henceforth we may assume that $Z_1 = \emptyset$ (and hence $Y = \emptyset$).

Subcase (iii) $|X| \geq 1$. Since $Y = \emptyset$ we have $M_{21} \cup M_{11} \neq \emptyset$. Choose $x_1x_2 \in M_{21} \cup M_{11}$ with $x_1 \in X_1$. We obtain a new 3-edge-colouring by deleting the edge x_1x_2 from S_0 and adding the edges ux_1, ux_2 coloured c_2 and c_3, respectively. Let $G =$

$G_1 + x_1x_2 - ux_1 - ux_2$. Then $\Delta_G = n-3$ and $G_\Delta = \{u\}$. Thus $q(G) = n-3$ by Fournier's Theorem and hence $q(G) \leq n$. Henceforth we may assume that $X = \emptyset$ and hence $S_0 = S$.

Subcase (iv) $|Z_2| \geq 1$. Choose $z \in Z_2$. Since $Z_1 = \emptyset$ and $S_0 = S$ we have $d_{G_1}(v) \leq n-5$ for all $v \in S$. Choose distinct edges $e_1, e_2 \in S$ containing z and suppose $\{z, x_1, y_1\} \subseteq e_1$ and $\{z, x_2, y_2\} \subseteq e_2$. Without loss of generality we may suppose that e_1, e_2 are coloured c_1, c_2 respectively. We modify the colouring of S by recolouring e_1 with a new colour c_4 and then colouring uz, ux_1, ux_2 with colours c_3, c_2, c_4, respectively. Let $G = G_1 - \{uz, ux_1, ux_2\}$. Then $\Delta_G = n-4$ and $G_\Delta = \{u\}$. Thus $q(G) = n-4$ by Fournier's Theorem and hence $q(G) \leq n$. Henceforth we may assume that $Z_2 = \emptyset$.

Subcase (v) $|Z_3| \geq 1$. Choose $z \in Z_3$. Since $Z_1 \cup Z_2 = \emptyset$ and $S_0 = S$ we have $d_{G_1}(v) \leq n-7$ for all $v \in S$. Choose distinct edges $e_1, e_2 \in S$ containing z and suppose $\{z, x_1, y_1\} \subseteq e_1$ and $\{z, x_2, y_2\} \subseteq e_2$. Without loss of generality we may suppose that e_1, e_2 are coloured c_1, c_2 respectively. We modify the colouring of S by recolouring e_1, e_2 with new colours c_4, c_5 and then colouring ux_1, uy_1, ux_2, uy_2 with colours c_1, c_5, c_2, c_4, respectively. Let $G = G_1 - \{ux_1, uy_1, ux_2, uy_2\}$. Then $\Delta_G = n-5$ and $G_\Delta = \{u\}$. Thus $q(G) = n-5$ by Fournier's Theorem and hence $q(G) \leq n$. This completes the proof of the lemma. □

3 Conclusion

We may deduce the following special case of Conjecture 3 from Lemmas 2.3, 2.4 and 2.5.

Theorem 3.1 *Let H be a loopless linear hypergraph on n vertices and let S be the partial hypergraph determined by the edges of size at least 3. If S has the edge-colouring property and $\Delta_S \leq 3$, then $q(H) \leq n$.*

Let H be a hypergraph and $V' \subseteq V(H)$. The subhypergraph H' of H with vertex set V' and edge set $E' = \{e_i \cap H' : 1 \leq i \leq m, e_i \cap V' \neq \emptyset\}$ is called the subhypergraph of H *induced by* V'. By duality, Theorem 3.1 gives the following special case of Conjecture 1.

Corollary 3.2 *Let H be a linear hypergraph consisting of n edges, each of size n, and let S be the partial hypergraph S of H induced by the vertices of degree at least 3. If $|e| \leq 3$, for all $e \in E(S)$, and S can be 3-coloured, then it is possible to colour the vertices of H with n colours so that no two vertices in the same edge receive the same colour.*

Several classes of hypergraphs that generalise bipartite graphs are known to have the edge-colouring property (see, for example, [1] Chapter 5). These include the class of unimodular hypergraphs. A matrix A is said to be *totally unimodular* if the determinant of each square submatrix of A has one of the values 0, 1 or -1. A hypergraph H is said to be *unimodular* if its incidence matrix is totally umimodular. It follows that the dual H^* of a hypergraph H is unimodular if and only if H is unimodular. We thus have the following particular cases of Theorem 3.1 and Corollary 3.2.

Corollary 3.3 *Conjecture 3 is true when the partial hypergraph S of H determined by the edges of size at least 3 is unimodular and satisfies $\Delta_S \leq 3$.*

Corollary 3.4 *Conjecture 1 is true when the subhypergraph S of H induced by the vertices of degree at least 3 is unimodular and such that $|e| \leq 3$, for all $e \in E(S)$.*

Finally, we note that there is a polynomial time algorithm developed by Bixby [3] to test whether a given hypergraph is unimodular.

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